math1090
Introduction to Logic for Computer Science

Lecture 3

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Recap

Inductive definition of sets

An inductive definition of a set consists of

1. **A universe set** U
2. **A core set** $C \subseteq U$
3. **A finite set** $O = \{o_1, o_2, \ldots o_n\}$ **of operations from** $o_i : U^{r_i} \rightarrow U$ **for some arities** $r_i \in \mathbb{N}$

We can also think of $I(U, C, O)$ as the set of elements that we obtain by starting with the core set and putting all those elements of U into $I(U, C, O)$ that one can reach by successively applying the operations in O.

Inductive definition of sets – another example

1. We'd like to define the set of even natural numbers. We choose:
   - Universe $U = \mathbb{R}$
   - Core set $C = \{2\}$
   - Set of operation $O = \{o_1\}$ where $o_1 : x \mapsto x + 2$.

   Then $I(U, C, O) = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\}$

The set WFF of well formed propositional formulas

The set WFF of well formed propositional formulas is the inductively defined set $I(\Sigma^*, P, O)$, where the three components are defined as follows:

1. **Universe:** Σ^*, the set of all strings over the alphabet of propositional logic
2. **Core set:** The set P of all propositional variables
3. **Operations:** The set $O = \{o_\neg, o_\wedge, o_\vee, o_\rightarrow\}$, defined as follows:
   - $o_\neg : \varphi \mapsto (\neg\varphi)$
   - $o_\wedge : \varphi, \psi \mapsto (\varphi \wedge \psi)$
   - $o_\vee : \varphi, \psi \mapsto (\varphi \vee \psi)$
   - $o_\rightarrow : \varphi, \psi \mapsto (\varphi \rightarrow \psi)$

Well Formed Formulas

How to decide if a string is a WFF?

We would like to have a way of determining whether some string $\alpha \in \Sigma^*$ is a well formed formula.

Are these in WFF?

- (p)
- $q \to p$
- $((p \vee q) \wedge (\neg p \wedge q))$
- $((p \to q) \wedge ((\neg p) \to q))$
- $(((p_1 \vee q_1) \wedge (\neg p_2 \wedge q_1)) \to ((p_2 \to q_2) \wedge ((\neg p_1) \to q_2))$

Showing that a formula is indeed in WFF

To prove that a string is indeed a well formed formula we can show how it was constructed.

This is formalized with the notion of a construction sequence (called formula calculation in textbook).

Construction sequence

A construction sequence for a formula $\alpha \in \mathrm{WFF}$ is a sequence of strings

$$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$$

for some $n \in \mathbb{N}$, such that each α_i is either a member of the coreset of WFF or it is the result of applying one of the operations $o_\neg, o_\wedge, o_\vee, o_\rightarrow$ to some α_j and α_k (or simply to α_j) where $j, k < i$ and such that $\alpha_n = \alpha$.

Construction sequence – example

To show that $\alpha = ((p \to q) \wedge ((\neg p) \to q))$ (the second to last example on the earlier slide) is indeed a well-formed formula, we can provide the following construction sequence as certificate:

$$\alpha_1 = p$$
$$\alpha_2 = q$$
$$\alpha_3 = (p \to q)$$
$$\alpha_4 = (\neg p)$$
$$\alpha_5 = ((\neg p) \to q)$$
$$\alpha_6 = ((p \to q) \wedge ((\neg p) \to q))$$

Task for you: Provide a justification for each line in the above construction sequence!

For example: we obtain the third line, by applying $o_\to$ to line 1 and line 2. That is $\alpha_3 = o_\to(\alpha_1, \alpha_2)$.

Construction sequences

- We have defined the notion of a construction sequence for the set of WFF of well formed formulas.

- WFF was an example of an inductively defined set $(\mathrm{WFF} = I(\Sigma^*, P, O))$.

- Construction sequences can be defined more generally for inductively defined sets, and one could then use them to certify membership in those sets.

Construction sequence – example

We have seen above how to define the set of even natural numbers inductively:

- Universe $U = \mathbb{R}$
- Core set $C = \{2\}$
- Set of operation $O = \{o_1\}$ where $o_1 : x \mapsto x + 2$.

Then $I(U, C, O) = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\}$

Now, we can, for example, show that 14 is indeed an even number by presenting a construction sequence for it:

$$2, 4, 6, 8, 10, 12, 14$$

Showing that a formula is not in WFF

- To show that some string α is not a well formed formula, we can not just say: "I can not find a construction sequence, so I believe it is not".

- We would like to have a more reliable argument.

- The idea is to identify some property, show that all strings in WFF satisfy this property, but the string in question does not.

- To show that all strings in WFF satisfy a certain property, we will use structural induction.

Structural induction–general definition

Consider some inductively defined set $\mathcal{A} = I(U, C, O)$. To show that all elements of $\mathcal{A}$ satisfy property $\mathcal{P}$ we prove the following:

Base case Show that all elements $c \in C$ of the core set satisfy the property

Induction hypothesis Assume that some $a_1, a_2, \ldots a_n$ satisfy the property
(here n needs to be the largest arity of the operations in O)

Induction step Show that (if the induction hypothesis holds) for all operation $o_i \in O$, the property also holds for

$$o_i(a_1, a_2, \ldots a_{r_i}).$$

Structural induction

- Structural induction is a general method to prove statements (properties) for inductively defined sets

- You may have seen the method in the case of proving statements for natural numbers

- When proving something by (structural) induction, it is very important that you clearly state the hypothesis and make it clear to yourself where in the induction step you are actually using it. If it is not clear where you use it, there is likely something wrong with your prove..!

Structural induction – definition for WFF

To show that all well formed formulas (elements of WFF) satisfy property $\mathcal{P}$ we prove the following:

Base case Show that all atomic formulas (that is formulas consisting only of a singe proposition variable) satisfy the property

Induction hypothesis Assume that two elements $\alpha_1, \alpha_2 \in \mathrm{WFF}$ satisfy the property

Induction step Show that the four following formulas

- $(\neg\alpha_1)$
- $(\alpha_1 \vee \alpha_2)$
- $(\alpha_1 \wedge \alpha_2)$
- $(\alpha_1 \rightarrow \alpha_2)$

also satisfy the property.