math1090
Introduction to Logic for Computer Science
Lecture 2

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September 10, 2018

Revision of lecture 1

Basic Information

- Instructor: Ruth Urner (ruth@eecs.yorku.ca)

- Office Hours: Tuesday 2-4pm or by appointment (LAS3016)

- Website: Currently being setup on http://moodle.yorku.ca
should be available soon

Lectures

Lecture times	Mon, Wed, 2:30pm - 4pm
Place	Mon: SLH F, Wed: DB 0016
First lecture	Wednesday, September 5
Last lecture	Monday, December 3
Reading week	October 7-12

Evaluation

Assignments:

- 5 Assignments
- Roughly every two weeks starting next week

Tests:

- In-class midterm (expect mid/end October)
- Final exam (December)

Mark composition:

- 1/3 % Assignments
- 1/3 % Midterm
- 1/3 % Final exam

Material

Textbook: George Tourlakis: Mathematical Logic.
(not mandatory)

Slides used in class will generally be posted to the course website within 24 hours after lecture.

Material in readings and lectures will overlap but won't be identical. Some material will only be presented on board.
You are responsible for both!

Don't skip lectures and plan to catch up by looking at the posted notes only! Ask your friends for notes.

Encouragement

If anything is unclear, ask questions!!!

Motivation

What is Logic about?

Logic provides a framework for how to reliably get from from assumptions to conclusions in reasoning.

It is **not** about distinguishing truth from falsehood.

What is Logic about? - examples

Assumptions:
- All people are mortal
- I am a person

(Sad) **Conclusion:**
- I am mortal

It is not up to the logician to decide on the correctness of the assumption (in this case, that may be up to biologists).
Logic is just concerned on whether the step of moving from the assumptions to the conclusion is a valid one.

What is Logic about? - examples

The previous example was a case of correct reasoning.
What about this one?

Assumptions:

- Leopards have dots
- I have dots

Conclusion:

- I am a leopard

This one obviously is not..

Do we really need a whole theory of logic to distinguish such cases?
Seems pretty obvious here..

Russell's paradox

In some cases however, it is not so clear to decide which conclusions are valid..

The barber's paradox
Consider the following statement: The barber of the village (a man himself) every morning shaves every man (and only those) in the village that does not shave themselves.

Question:
Does the barber shave himself?

Here, we can argue that the barber shaving himself implies that he does not shave himself and vice versa..

- Make it clear to yourself that you get these contradicting conclusions!
- To resolve this paradox: realize that, due to the contradition, no such village exists.

Goal

We need to develop reliable tools for going from assumptions to implied conclusions.

First observation
We can not rely on English (or any other “natural” language) for this purpose.

Natural languages contain lots of ambiguities.

For example, in English, we can prove that a hotdog is better than eternal happiness..

Ambiguities in natural language – example

Consider the following reasoning pattern:

Assumptions:

- 95 is better (as a mark for this course) than 70
- 70 is better than 40

Conclusion:

- 95 is better than 40.

Now, let's apply the same pattern to hotdogs and happiness:

Assumptions:

- A hotdog is better than nothing
- Nothing is better than eternal happiness

Conclusion:

- A hotdog is better than eternal happiness

Ambiguities in natural language – example

- A hotdog is better than eternal happiness...?

Does not seem right...(although the assumptions sounded fine, and we applied the exact same pattern for reasoning as in the marks example).

The problem here is that the word “nothing” is used in two different ways. Natural languages contain lots of such ambiguities.

Our goal is now to develop a formal language and ways to go from assumptions to conclusions in this formal framework.

We start with the language of propositional logic. Later we will move on to predicate logic (also called first oder logic).

An important distinction

In logic, we separate out the syntax (form) and the semantics (meaning) of our formal system.

This is a very important distinction keep in mind all the time.

Our goal is to first develop the system purely on the syntax side (that is, we decide what are the formal words in our new language, how to make derivations in this system purely according to some formal rules etc) and then show that this coincides with what we intended derivations in such a system to mean.

Propositional logic

Syntax of propositional logic

We start with defining the syntax of the language of propositional logic.

Our first goal is to define what are "acceptable" statements in this language. These will form the set of well formed formulas.

How to define sets

Let's recall some tools we know for defining sets of objects:

1. Make a list of the elements
   - Set of all students in this class
   - Set of odd natural numbers smaller than 10: $\{1, 3, 5, 7, 9\}$

   Problem: this technique fails for large or infinite sets
2. Identify by a common characteristic
   - Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is not divisible by } 2\}$

   Problem: sometimes we don't know a precise defining characteristic
3. Inductive definition
   - We'll see how to do this next!

How to define sets – a warning

Warning:
When we define sets, we always need to specify from which universe (that is a possibly much larger ground set) the elements of our set should be taken from!

Example:
- Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is divisible by } 2\}$
- Interval on the real line $\{x \in \mathbb{R} \mid 2 \leq x \leq 4\}$

Otherwise we can fall back into Russell's paradox...!

How to define – a warning

If we are not careful about specifying the universe from which we take elements when defining a new set, we can get back into paradoxical situations.

Here is the "original version" of Russell's paradox – a paradox in Mathematics.
(Try to see how this is similar to the barber's paradox version that we've seen earlier.)

Consider the following definition of a set:

$$R = \{r | r \notin r\}$$

That is, the set R contains all those sets that do not contain themselves as an element.

Question: Is R an element of R?

Now, $R \in R$ implies that $R \notin R$, and vice versa ($R \notin R$ implies that $R \in R$)

– a contradiction.

Inductive definition of sets – example

Say, I'd like to define the set of all (biological) relatives of mine (living and dead ones).

- **I can not make a list**
  (I don't know them all, especially not those that lived a thousand years ago..)
- **I can not give a precise characteristic**
  (Maybe I could if I was a biologist, but I am not..)
- **But I know some operations that will allow me to get from me to all of them!**
  The idea is to start with me, and consider everyone that can be reached by successively considering all children and all parents of previously reached people.

Inductive definition of sets – example

Here is a more formal way of defining all my biological relatives:

Consider the following three ingredients:

1. **Universe:** all people
2. **Core set:** me
3. **Operations:** parent-of, child-of

Now the set of all my relatives is the smallest set of people that contains me and is closed under the stated operations.

Another way of thinking about it: Start with me, and successively apply the operations parent-of and child-of.

Inductive definition of sets

An inductive definition of a set consists of

1. **A universe set** U
2. **A core set** $C \subseteq U$
3. **A finite set** $O = \{o_1, o_2, \ldots o_n\}$ **of operations (functions)** $o_i : U^{r_i} \rightarrow U$ **for some arities** $r_i \in \mathbb{N}$

The set $I(U, C, O)$ defined inductively by these three components is the smallest subset of U that contains the coreset C and is closed under the operations in O.

Aside: A set being closed under operations

A set $A \subseteq U$ is closed under some set of operations $O = \{o_1, o_2, \ldots o_n\}$ if, for all $o_i \in O$ and all $a_1, a_2, \ldots a_{r_i} \in A$ we have

$$o_i(a_1, a_2, \ldots a_{r_i}) \in A$$

Aside: A set being closed under operations–examples

1. Universe $\mathbb{R}$
   Set $\mathbb{N}$
   Operation $+ : \mathbb{R}^2 \to \mathbb{R}$, $+ : (x, z) \mapsto x + z$.
   When we add two natural numbers, the result is again a natural number.
   Thus $\mathbb{N}$ is **closed** under $+$.
2. Universe $\mathbb{R}$
   Set $\mathbb{N}$
   Operation $- : \mathbb{R}^2 \to \mathbb{R}$, $- : (x, z) \mapsto x - z$.
   When we substract a natural from another natural number, the result can be a negative integer, and hence not a natural number.
   Thus $\mathbb{N}$ is **not closed** under $-$.
3. Universe $\mathbb{R}$
   Set $\mathbb{Z}$
   Operation $- : \mathbb{R}^2 \to \mathbb{R}$, $- : (x, z) \mapsto x - z$.
   When we substract an integer from another integer, the result is an integer.
   Thus $\mathbb{Z}$ is **closed** under $-$.

Inductive definition of sets – alternative way of thinking of it

An inductive definition of a set consists of

1. **A universe set** U
2. **A core set** $C \subseteq U$
3. **A finite set** $O = \{o_1, o_2, \ldots o_n\}$ **of operations from** $o_i : U^{r_i} \to U$ **for some arities** $r_i \in \mathbb{N}$

We can also think of $I(U, C, O)$ as the set of elements that we obtain by starting with the core set and putting all those elements of U into $I(U, C, O)$ that one can reach by successively applying the operations in O.

Inductive definition of sets – examples

1. We'd like to define the set of natural numbers. We choose:
   - Universe $U = \mathbb{R}$
   - Core set $C = \{1\}$
   - Set of operation $O = \{o_1\}$ where $o_1 : x \mapsto x + 1$.

   Then $I(U, C, O) = \mathbb{N}$

2. We'd like to define the set of integers. We choose:
   - Universe $U = \mathbb{R}$
   - Core set $C = \{0\}$
   - Set of operation $O = \{o_1, o_2\}$ where $o_1 : x \mapsto x + 1$ and $o_2 : x \mapsto x - 1$.

   Then $I(U, C, O) = \mathbb{Z}$

The alphabet of propositional logic

Our goal is still to define the set of all well-formed formulas of propositional logic. Our universe will be the set of all words (strings) over the following alphabet Σ:

1. Propositional variables: $p, q, p_1, p_2, \ldots, q_1, q_2, \ldots$
   (lower case letters, possibly indexed)
2. Logical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$
   (We'll call them corner, wedge, vee, and arrow)
3. Punctuation (,)
   (opening and closing brackets)

Our universe is the set Σ^*, that is all finite sequences that can be built from the above symbols of the alphabet Σ.

The set WFF of well formed propositional formulas

The set WFF of well formed propositional formulas is the inductively defined set $I(\Sigma^*, P, O)$, where the three components are defined as follows:

1. **Universe:** Σ^*, the set of all strings over the alphabet of propositional logic
2. **Core set:** The set P of all propositional variables
3. **Operations:** The set $O = \{o_\neg, o_\wedge, o_\vee, o_\rightarrow\}$, defined as follows:
   - $o_\neg : \varphi \mapsto (\neg\varphi)$
   - $o_\wedge : \varphi, \psi \mapsto (\varphi \wedge \psi)$
   - $o_\vee : \varphi, \psi \mapsto (\varphi \vee \psi)$
   - $o_\rightarrow : \varphi, \psi \mapsto (\varphi \rightarrow \psi)$