Finite-lattice Hamiltonian results for the phase structure of the $Z(q)$ models and the q-state Potts models

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We use our finite-lattice approach to study the phase transitions of the Hamiltonian formulation for the $(1+1)$-dimensional $Z(q)$ models. We confirm the known result that these models possess two phases for $q < q_c$ and three phases for $q \geqslant q_c$. Our calculation, however, gives $q_c = 6$, while the perturbative calculation predicts $q_c = 5$. Similar calculations for the $(1+1)$-dimensional q-state Potts models for $q = 4$, 5, and 8 failed to differentiate the second-order transition expected for $q \leqslant 4$ from the first-order transition expected for $q \geqslant 5$.

I. INTRODUCTION

The phase structure of the $Z(q)$ models has attracted the attention of physicists for a long time. Using perturbation theory and Padé extrapolants, Elitzur *et al.*,[1] argued that the Hamiltonian formulation of the $Z(q)$ models exhibits a conventional Ising-like transition for $q < q_c = 5$, while for $q \geqslant q_c$, a massless phase separates the low- and high-temperature phases. Recently Rujan *et al.*[2] approached the problem using variational and Migdal recursion-relation techniques, and found that $q_c = 6$.

In this paper we apply our finite-lattice approach to the Hamiltonian form for these models. The finite-lattice calculational techniques are explained in Ref. 3. The extrapolation to the infinite-lattice limit is presented in Ref. 4. In Refs. 3 and 4 the finite-lattice approach was applied to a variety of models among which were the $q = 2$, 3, and ∞ $Z(q)$ models (the $q = 4$ model is nothing but a decoupled set of two $q = 2$ models). We obtained conventional second-order transitions for $q = 2$, 3, and for $q = \infty$, a line of fixed points extending from $x = x^*$ to ∞. It is very interesting to see how this line will build up, within our finite-lattice approach, as $q \to \infty$.

In Sec. IV of this paper we report similar calculations for the q-state Potts models for $q = 4$, 5, and 8. We obtain similar results for the three cases and these results are typical of second-order phase transitions. There is no sign in our calculations of the first-order phase transition expected for $q \geqslant 5$.

II. $Z(q)$ MODEL

The Hamiltonian of the $Z(q)$ model is given by[1]

$$H = (g/2a)W \ , \tag{1}$$

$$W = \sum_{i=1}^{N} \left[\frac{1-\cos(2\pi/q)L_i}{1-\cos(2\pi/q)} - x\cos\left[\frac{2\pi}{q}(M_i - M_{i+1})\right]\right] \ , \tag{2}$$

where $x = 2/g^2$, a is the lattice spacing, and L_i, M_i satisfy

$$L_{i+N}, M_{i+N} = L_i, M_i \ , \tag{3}$$

$$[L_i, M_j] = -i(q/2\pi)\delta_{ij} \ . \tag{4}$$

The coupling constant x is related to λ of Refs. 1 and 2 via

$$x = \lambda/[1-\cos(2\pi/q)] \ . \tag{5}$$

Recall that the theory is self-dual in λ, i.e., the mass gap satisfies $m(\lambda) = \lambda m(\lambda^{-1})$.

III. RESULTS FOR THE $Z(q)$ MODELS

According to the renormalization-group interpretation of Ref. 4, the quantity of interest is

$$G_H(x,N) = gNG(x,N) \ , \tag{6}$$

where $G(x,N)$ is the mass gap of W. At a fixed point $x = x^*$, the quantities $G_H(x^*,N)$ become equal for different N.

In Fig. 1, G_H for the $q = 5$ model, is plotted against x for $N = 2-6$. Figures 2, 3, and 4 are similar to Fig. 1 but for the $q = 6$, 7, and 9 models, respectively.

It should be noted that our calculation of G is not valid when x is very large. In that region the mass gap calculated above becomes identically zero when $N \to \infty$. (This region is the analog of the spontaneous magnetization phase of the Ising model.) The relevant mass gap of that region involves new sectors characterized by rather complicated topological excitations. (It is difficult to represent a single domain wall, for example, with periodic boundary conditions.) We did not compute the mass gap for these sectors, but since the transition character is not affected by that region of large x, the above observation does not affect our conclusions regarding the nature of the transition. We cannot, however, discuss the

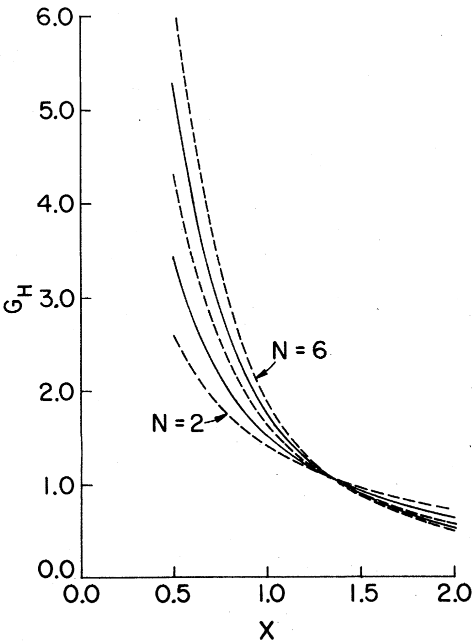

FIG. 1. Quantities $G_H(x,N)$ given by Eq. (6) vs the coupling parameter x defined in Eq. (2) for the $q=5$ model $Z(5)$. Separate curves are given for $N=2$–6. Note that the curves for $N=3$–6 appear to cross at $x \approx 1.4$.

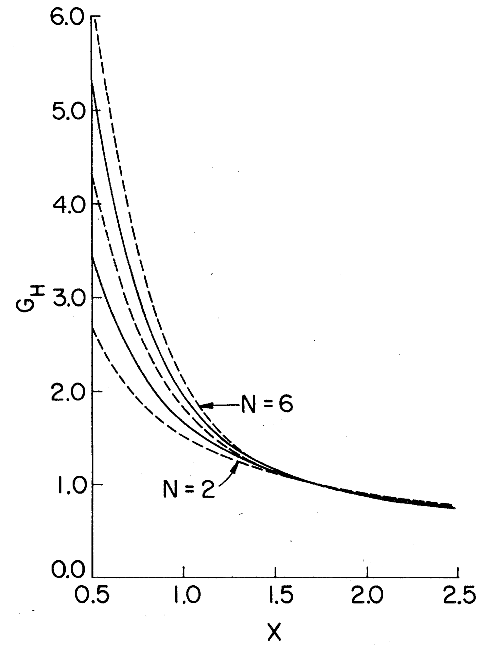

FIG. 3. Same as Fig. 1 for the $q=7$ model $Z(7)$. The curves for $N=3$–6 appear to meet at $x \approx 1.8$.

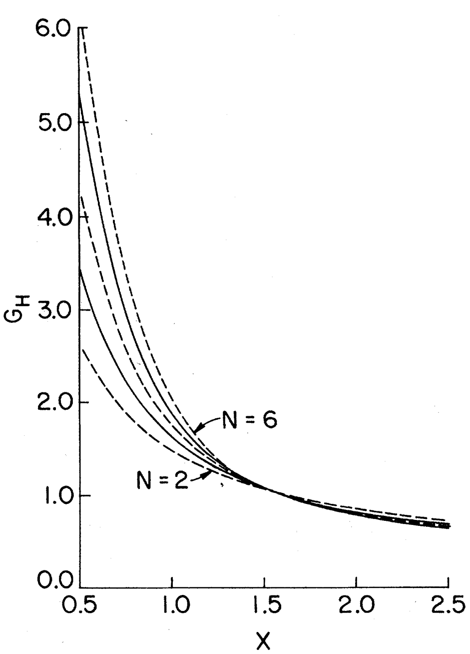

FIG. 2. Same as Fig. 1 for the $q=6$ model $Z(6)$. The curves for $N=3$–6 appear to meet at $x \approx 1.7$.

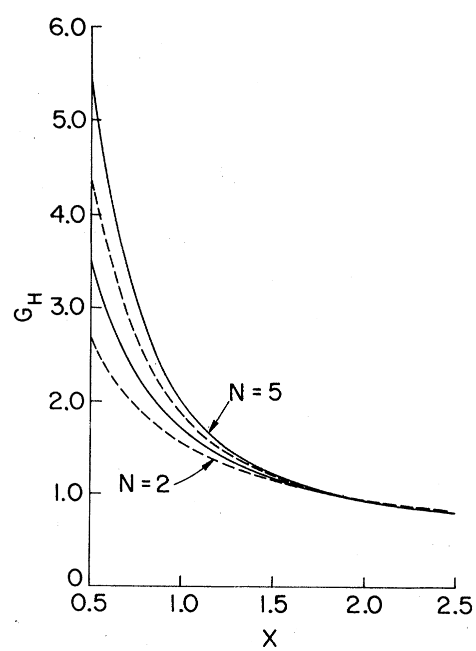

FIG. 4. Same as Fig. 1 for the $q=9$ model $Z(9)$. The curves for $N=3$–6 appear to meet at $x \approx 2.0$.

phase transition at the low-temperature end of a line of fixed points.

We see in Fig. 1 that the $Z(5)$ model has a conventional phase transition [the curves $G_H(x,N)$ intersect at one value of $x=x^*$], whereas Figs. 3 and 4 indicate the presence of a line of fixed points separating the high- and low-temperature phases. The $Z(6)$ model appears to be on the border line between these two regimes.

These conclusions can be obtained quantitatively by computing the β functions of these models. We use the formula

$$\frac{\beta(g)}{g}=\frac{\ln[N'G(g,N')/NG(g,N)]}{(\ln N'/N)\{1+\frac{1}{2}g(\partial/\partial g)\ln[NG(g,N)N'G(g,N')]\}} \quad (7)$$

derived in Ref. 4, to calculate $\beta(g)$. Here N and N' define the renormalization-group transformation $N=L/a \rightarrow N'=L/a'$, with L a fixed length and a, a' two lattice spacings. The numerical values obtained in this way are fitted to the form

$$\beta(g)=(1/c)(g-g^*)^{1+\sigma} \quad (8)$$

in the neighborhood of the critical point. If the fit gives $\sigma=0$, the transition is conventional second order and the correlation length index ν is given by $\nu=c$. For nonzero σ there is a line of fixed points.

In Table I, we summarize the results we obtained by fitting to Eq. (8) for the $q=5$–7 and 9 models. We see that for the $q=5$ case, $\sigma=0$ in agreement with the qualitative behavior of Fig. 1. For $q=6$, 7, and 9 on the other hand, we obtain a nonvanishing value for σ which means that the fixed-line regime sets in for $q \geq q_c=6$. These results differ from the perturbative results given in Sec. VI (Table III) of the paper by Elitzur, Pearson, and Shigemitsu (Ref. 1). Working with the same quantum Hamiltonian (2) and using Padé extrapolated strong-coupling expansions, these latter authors obtained $q_c=5$ and values for σ considerably larger than those given in our Table I.

Our result, along with the two methods of Ref. 2 (Migdal recursion relations and variational calculations), constitute a set of three independent approaches all leading to the result $q_c=6$ (if only integral q_c are considered).

TABLE I. The fixed point x^*, the correlation length index ν, and the index σ are shown for the $Z(q)$ models with $q=5$–7 and 9. When σ is nozero, the index ν is, of course, nonexistent.

q	x^*	ν	σ
5	1.40	2.4	0.0
6	1.67	· · ·	0.02
7	1.81	· · ·	0.06
9	2.00	· · ·	0.1

IV. q-STATE POTTS MODELS

The transfer matrix for the q-state Potts models was derived by Mittag and Stephen.[5] From this calculation one infers a Hamiltonian, which was put into a convenient form by Sólyom[6]:

$$H=(g/2a)W, \quad x=2/g^2 , \quad (9)$$

$$W=-\sum_{i=1}^{N}R_i - x\sum_{i=1}^{N}\sum_{k=1}^{q-1}M_i^kM_{i+1}^{q-k} , \quad (10)$$

with R_i and M_i the following $q\times q$ matrices for each site

$$R=\begin{pmatrix} q-1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & -1 \end{pmatrix} , \quad (11)$$

$$M=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \cdot & \\ & & & \cdot & \\ 0 & & & \cdot & 1 \\ 1 & 0 & & & 0 \end{pmatrix} . \quad (12)$$

From the point of view of the numerical calculation, W for the Potts models, Eq. (10), is very similar to W for the $Z(q)$ models, Eq. (2), and the computer program required only minor changes.

We were primarily interested in the Potts models because according to Baxter's theorem[7] the q-state Potts model has a second-order phase transition for $q \leq 4$ and a first-order phase transition for $q \geq 5$. Thus we wanted to see if our method, based on finite scaling of exact numerical results for small systems, could detect the first-order phase transition. In either case $q \leq 4$ or ≥ 5 duality[5,6] predicts that the phase transition for the W of Eq. (10) should occur at $x = 1$.

We performed our finite-lattice calculations for 5–7 sites for the two cases $q = 4$ and 5 and 4–6 sites for the case $q = 8$. The case $q = 8$ with 6 sites was the biggest calculation we could conveniently handle. In Fig. 5 G_H for the $q = 4$ model is plotted versus x for $N = 5$–7. Figure 6 is a similar plot for the $q = 5$ model, and Fig. 7 is a similar plot for the $q = 8$ model, except with $N = 4$–6 instead of 5–7. [Note that we use the same definitions as for the $Z(q)$ models: $H = (g/2a) W$, $x = 2/g^2$, and $G_H = gNG$ with G the mass gap of W.] We see that there are fixed points in the quantities G_H and hence phase transitions at $x \approx 1.0$ for all cases $q = 4$, 5, and 8. The three cases appear remarkably similar and have the general appearance of a second-order phase transition. Figures 5–7 are very similar in appearance to the corresponding plots generated for the $(2+1)$-dimensional Ising model and recorded in Ref. 4.

Thus our finite-lattice approach can easily find the phase transition at $x = 1.0$, but in its present form, appears unable to differentiate between first- and second-order transitions, at least for the Potts Hamiltonian. Both cases $q \geq 5$ and ≤ 4 lead to isolated fixed points with a gap G which scales as N^{-1}, for the values of N for which we were able to do the calculation. It should be emphasized that this is not a de-

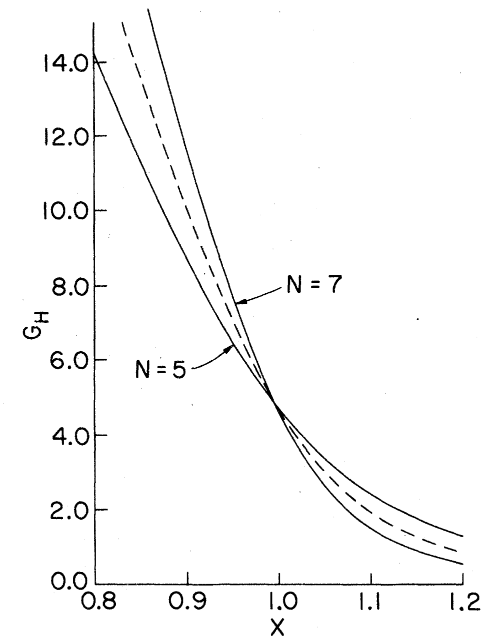

FIG. 6. Quantity $G_H(x,N)$ defined in Eq. (6) for the Potts Hamiltonian (9,10) for the case $q = 5$. The three curves correspond to $N = 5$–7 sites.

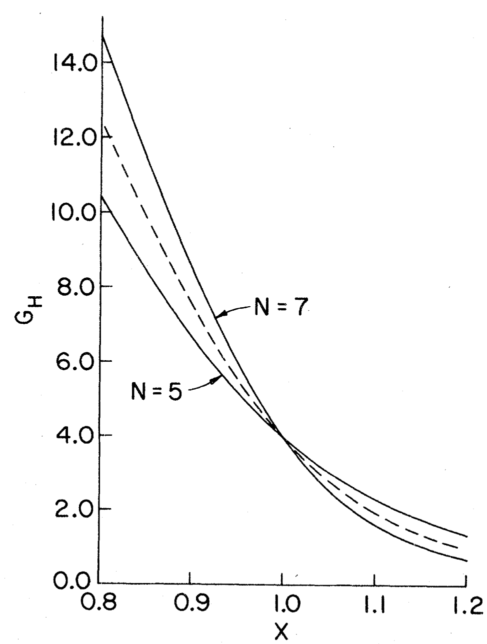

FIG. 5. Quantity $G_H(x,N)$ defined in Eq. (6) for the Potts Hamiltonian (9,10) for the case $q = 4$. The three curves correspond to $N = 5$–7 sites.

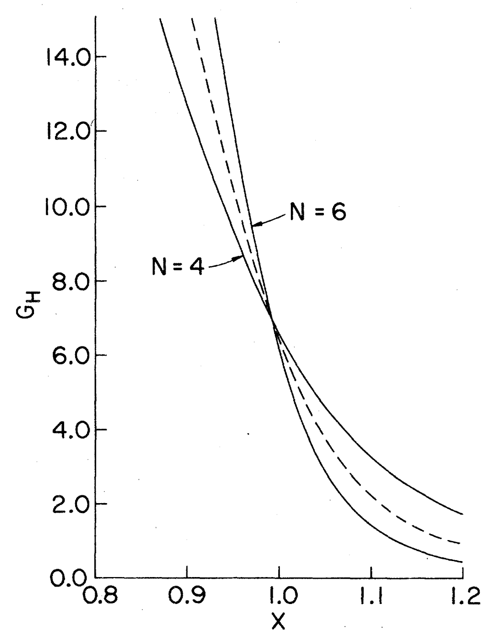

FIG. 7. Quantity $G_H(x,N)$ defined in Eq. (6) for the Potts Hamiltonian (9,10) for the case $q = 8$. The three curves correspond to $N = 4$–6 sites.

fect of the numerical method; the numerical results obtained are exact for the small values of $N \lesssim 7$, for which we could do the calculation. The difficulty lies with scaling the results to the $N \rightarrow \infty$ system of interest. The correlation length in lattice units is $l/a = 2/gG$. For a second-order transition $G \rightarrow 0$ and $l/a \rightarrow \infty$; in fact $G \sim \text{const}/N$. For a first-order transition $G \rightarrow \text{const}$, leading to finite l/a. However if l becomes larger than the finite-lattice size Na, the finite lattice "thinks" l is infinite and produces results similar to those for a second-order transition.

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