# EECS6414: Data Analytics & Visualization

# Network Properties: Characterizing/ Measuring Networks

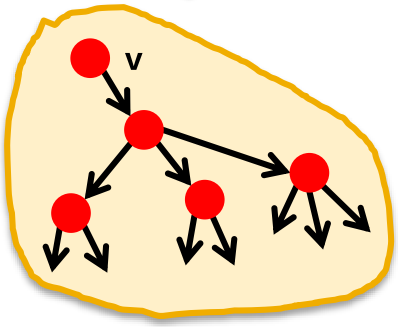
Thanks to Jure Leskovec, Stanford & Panayiotis Tsaparas, Univ. of Ioannina for slides

# Agenda

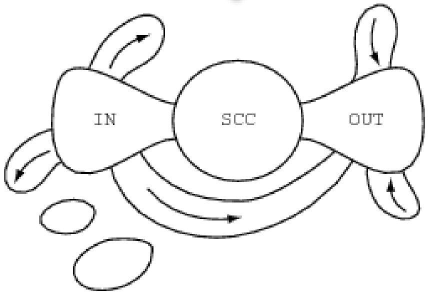
- Characterizing/Measuring Networks
  - Network Properties
- Case Study: A Real World Network (MSN)

# Structure of Networks

- **For example, last time we talked about Observations and Models for the Web graph:**
  - **1)** We took a real system: **the Web**
  - **2)** We represented it as a **directed graph**
  - **3)** We used the language of graph theory
    - **Strongly Connected Components**
  - **4)** We designed a **computational experiment:**
    - Find In- and Out-components of a given node $v$
  - **5) We learned something about the structure of the Web: BOWTIE!**



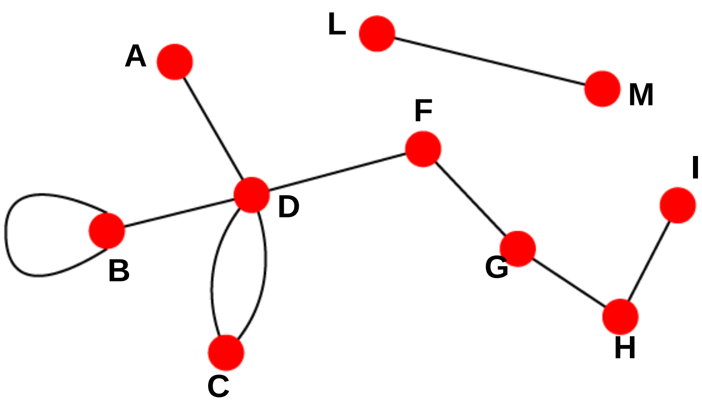

*Out(v)*



# Undirected vs. Directed Networks

## Undirected graphs

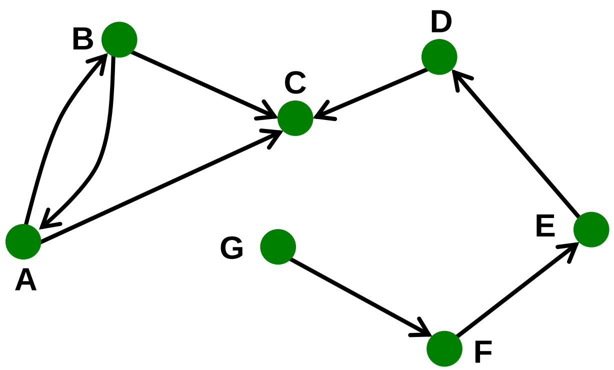
- **Links:** undirected (symmetrical, reciprocal relations)




- Undirected links:
  - Collaborations
  - Friendship on Facebook

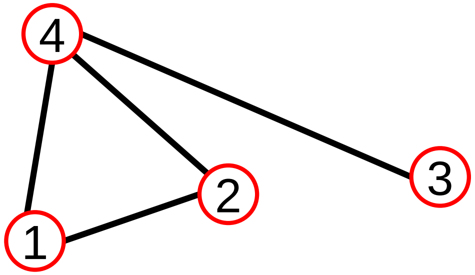
## Directed graphs

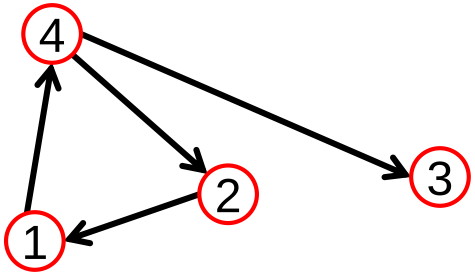
- **Links:** directed (asymmetrical relations)




- Directed links:
  - Phone calls
  - Following on Twitter

# Adjacency Matrix




$A_{ij} = 1$ if there is a link from node ***i*** to node ***j***

$A_{ij} = 0$ otherwise

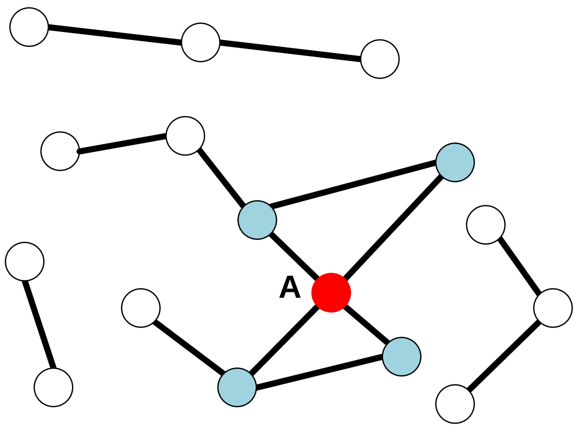
$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

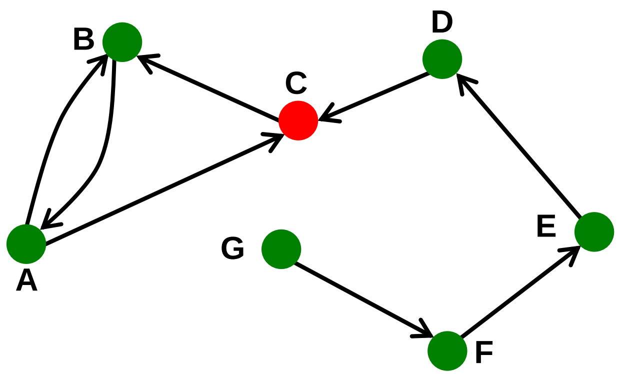
Note that for a directed graph (right) the matrix is not symmetric.

# Node Degrees

Undirected




Directed




**Source:** Node with $k^{in} = 0$

**Sink:** Node with $k^{out} = 0$

**Node degree, $k_i$:** the number of edges adjacent to node $\boldsymbol{i}$

$$k_A = 4$$

**Avg. degree:** $\bar{k} = \langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2E}{N}$

In directed networks we define an **in-degree** and **out-degree.**

The (total) degree of a node is the sum of in- and out-degrees.

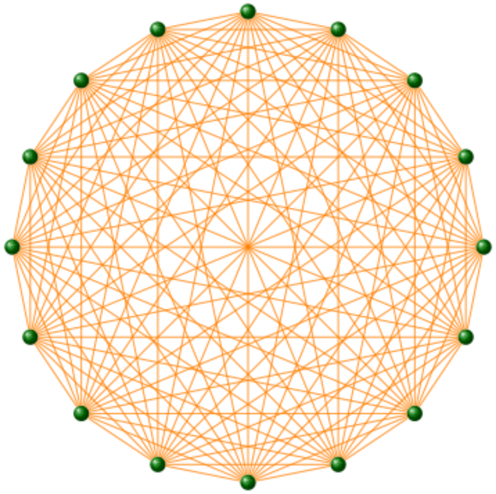
$$k_C^{in} = 2 \qquad k_C^{out} = 1 \qquad k_C = 3$$

$$\bar{k} = \frac{E}{N} \qquad \overline{k^{in}} = \overline{k^{out}}$$

# Complete Graph

The **maximum number of edges** in an undirected graph on $N$ nodes is

$$E_{\max} = \binom{N}{2} = \frac{N(N-1)}{2}$$



An undirected graph with the number of edges $E = E_{max}$ is called a **complete graph**, and its average degree is $N\text{-}1$

# Networks are Sparse Graphs

## Most real-world networks are sparse

$$E \ll E_{max} \quad (\text{or } \bar{k} \ll N-1)$$

| | | |
|---|---|---|
| WWW (Stanford-Berkeley): | N=319,717 | $\langle k \rangle$=9.65 |
| Social networks (LinkedIn): | N=6,946,668 | $\langle k \rangle$=8.87 |
| Communication (MSN IM): | N=242,720,596 | $\langle k \rangle$=11.1 |
| Coauthorships (DBLP): | N=317,080 | $\langle k \rangle$=6.62 |
| Internet (AS-Skitter): | N=1,719,037 | $\langle k \rangle$=14.91 |
| Roads (California): | N=1,957,027 | $\langle k \rangle$=2.82 |
| Proteins (S. Cerevisiae): | N=1,870 | $\langle k \rangle$=2.39 |

(Source: *Leskovec et al., Internet Mathematics, 2009*)

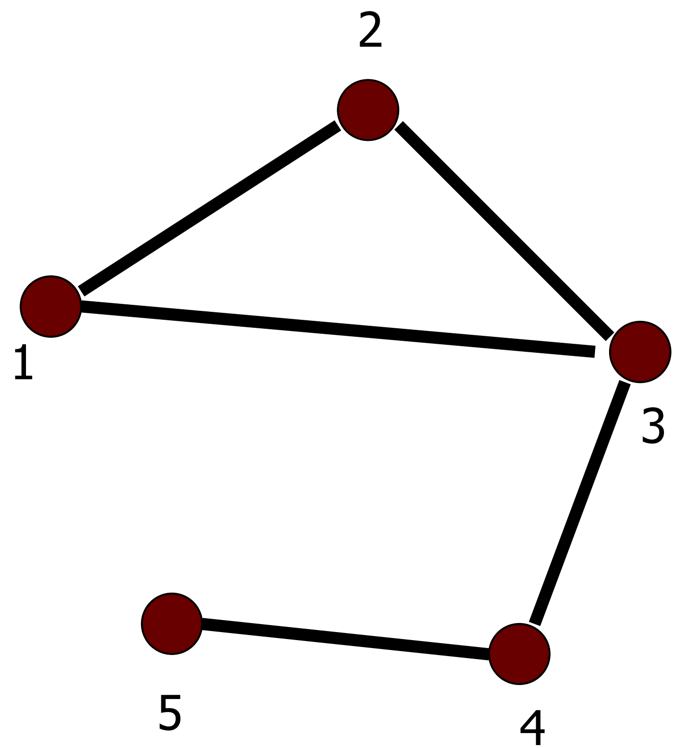
**Consequence: Adjacency matrix is filled with zeros!**

(**Density of the matrix ($E/N^2$):** WWW=$1.51\times10^{-5}$, MSN IM = $2.27\times10^{-8}$)

# Graph Representation

- Adjacency Matrix
  - symmetric matrix for undirected graphs

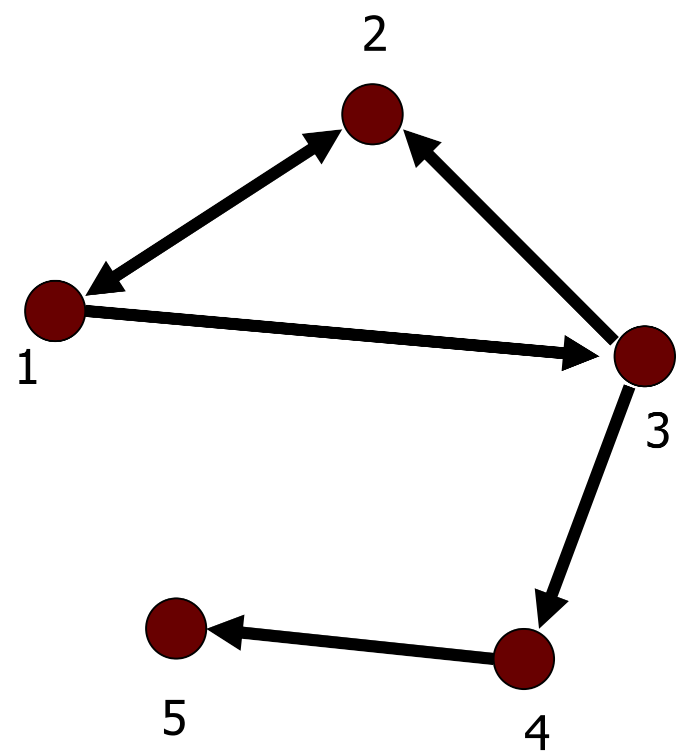
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



# Graph Representation

- Adjacency Matrix
  - unsymmetric matrix for undirected graphs

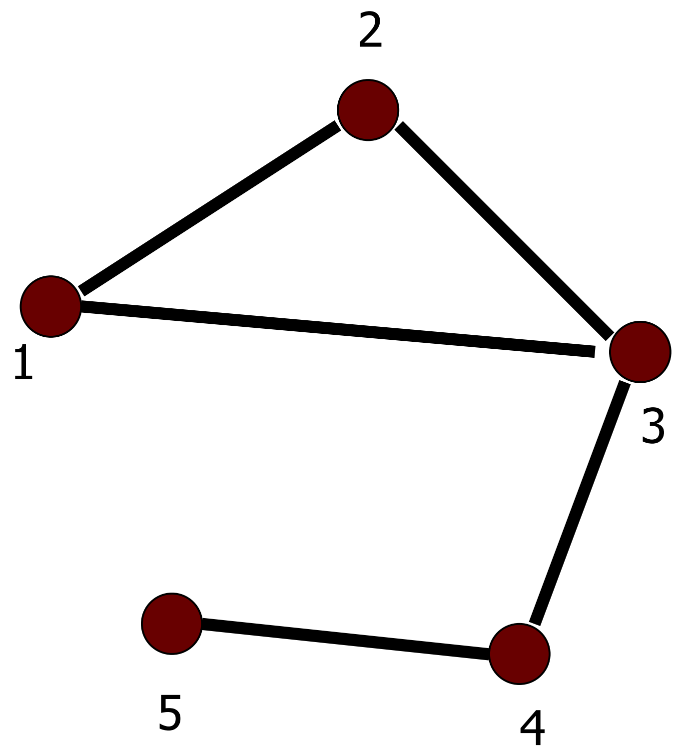
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



# Graph Representation

- Adjacency List
  - For each node keep a list with neighboring nodes

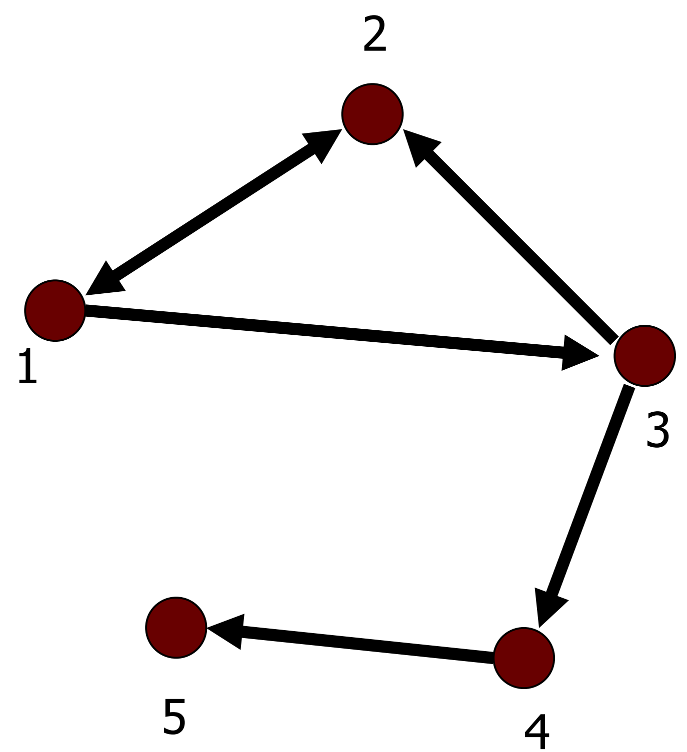
1: [2, 3]
2: [1, 3]
3: [1, 2, 4]
4: [3, 5]
5: [4]



# Graph Representation

- Adjacency List
  - For each node keep a list of the nodes it points to

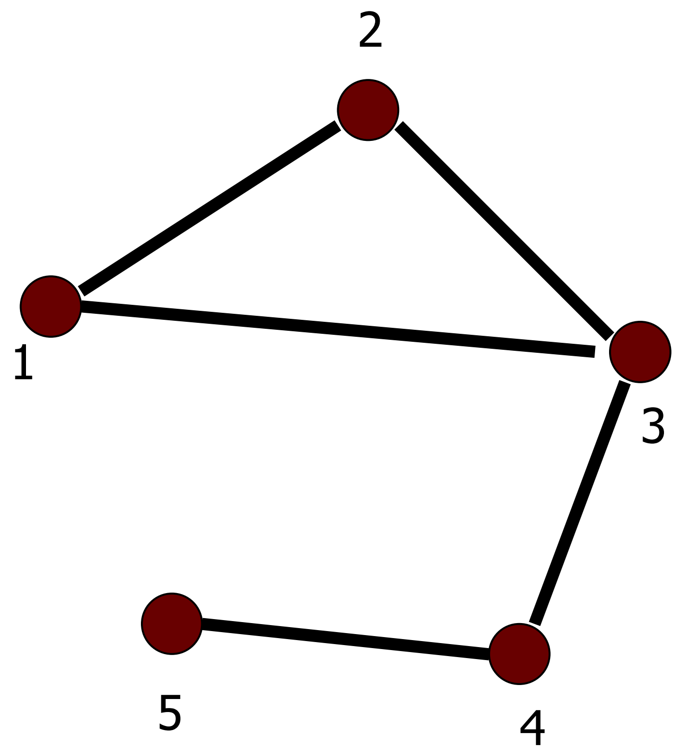
1: [2, 3]
2: [1]
3: [2, 4]
4: [5]
5: [null]



# Graph Representation

- List of edges
  - Keep a list of all the edges in the graph

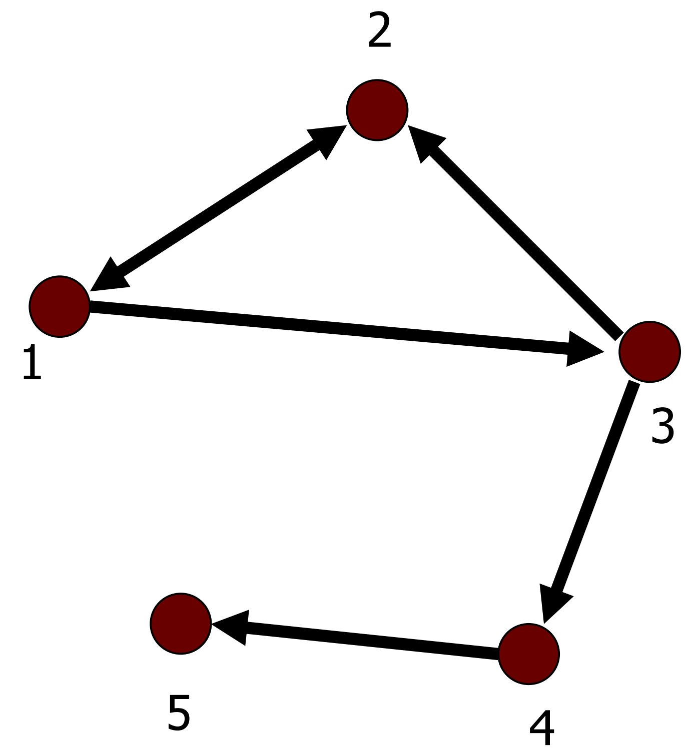
(1,2)
(2,3)
(1,3)
(3,4)
(4,5)



# Graph Representation

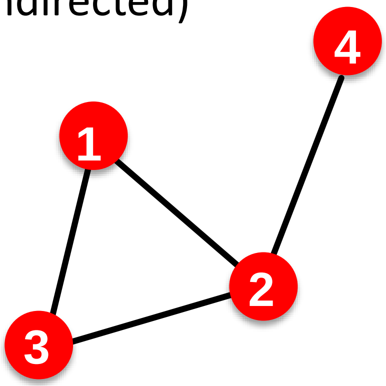
- List of edges
  - Keep a list of all the directed edges in the graph

(1,2)
(2,1)
(1,3)
(3,2)
(3,4)
(4,5)



# More Types of Graphs:

- **Unweighted** (undirected)



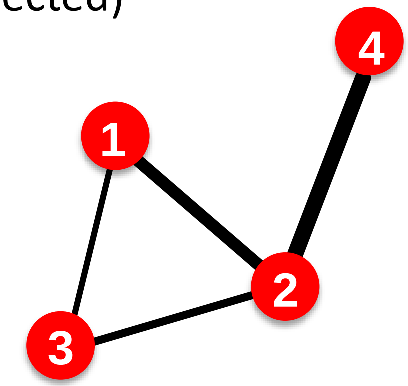

$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij} \qquad \bar{k} = \frac{2E}{N}$$

**Examples:** Friendship, Hyperlink

- **Weighted** (undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 0.5 & 0 \\ 2 & 0 & 1 & 4 \\ 0.5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

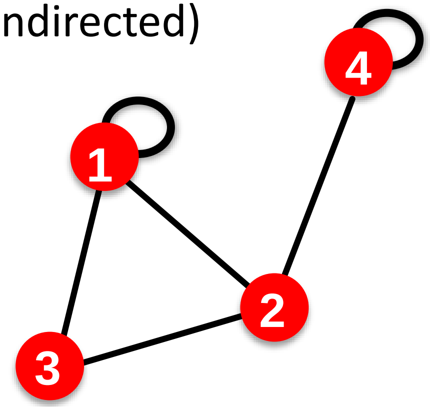
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

**Examples:** Collaboration, Internet, Roads

# More Types of Graphs:

- **Self-edges (self-loops)**
  (undirected)



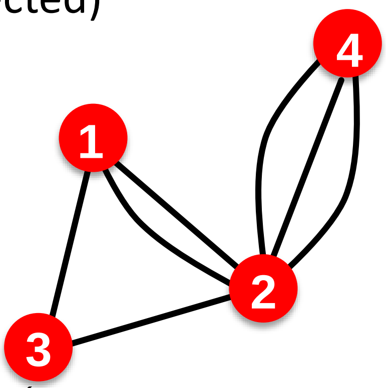

$$A_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_{ii} \neq 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} A_{ij} + \sum_{i=1}^{N} A_{ii}$$

**Examples:** Proteins, Hyperlinks

- **Multigraph**
  (undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

**Examples:** Communication, Collaboration

# Network Representations

WWW >> directed multigraph with self-edges

Facebook friendships >> undirected, unweighted

Citation networks >> unweighted, directed, acyclic

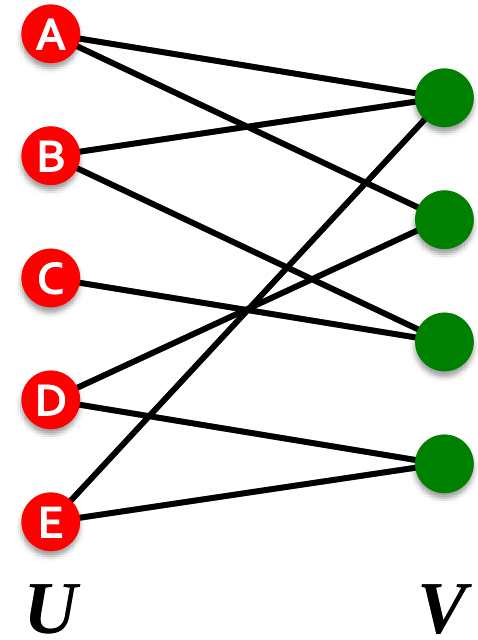
Collaboration networks >> undirected multigraph or weighted graph

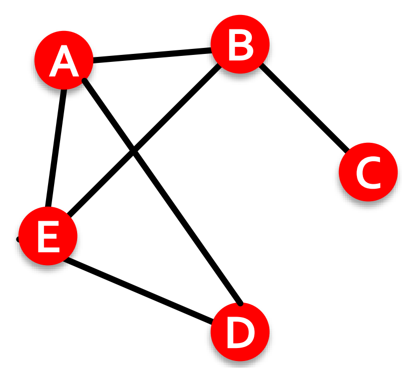
Mobile phone calls >> directed, (weighted?) multigraph

Protein Interactions >> undirected, unweighted with self-interactions

# Bipartite Graph

- **Bipartite graph** is a graph whose nodes can be divided into two disjoint sets $U$ and $V$ such that every link connects a node in $U$ to one in $V$; that is, $U$ and $V$ are **independent sets**
- **Examples:**
  - Authors-to-papers (they authored)
  - Actors-to-Movies (they appeared in)
  - Users-to-Movies (they rated)
- **"Folded" networks:**
  - Author collaboration networks
  - Movie co-rating networks

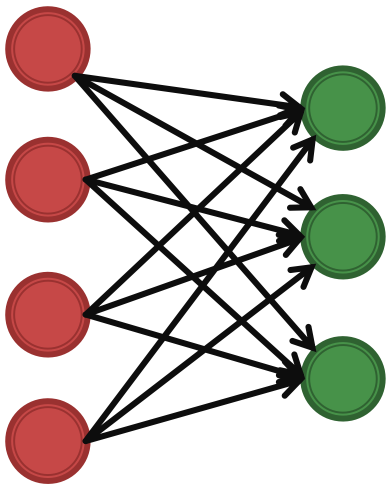



Folded version of the graph above

# Web Cores

- Cores: Small complete bipartite graphs (of size 3x3, 4x3, 4x4)
  - Similar to the triangles in undirected graphs
- Found more frequently than expected on the Web graph
- Correspond to communities of enthusiasts (e.g., fans of japanese rock bands)

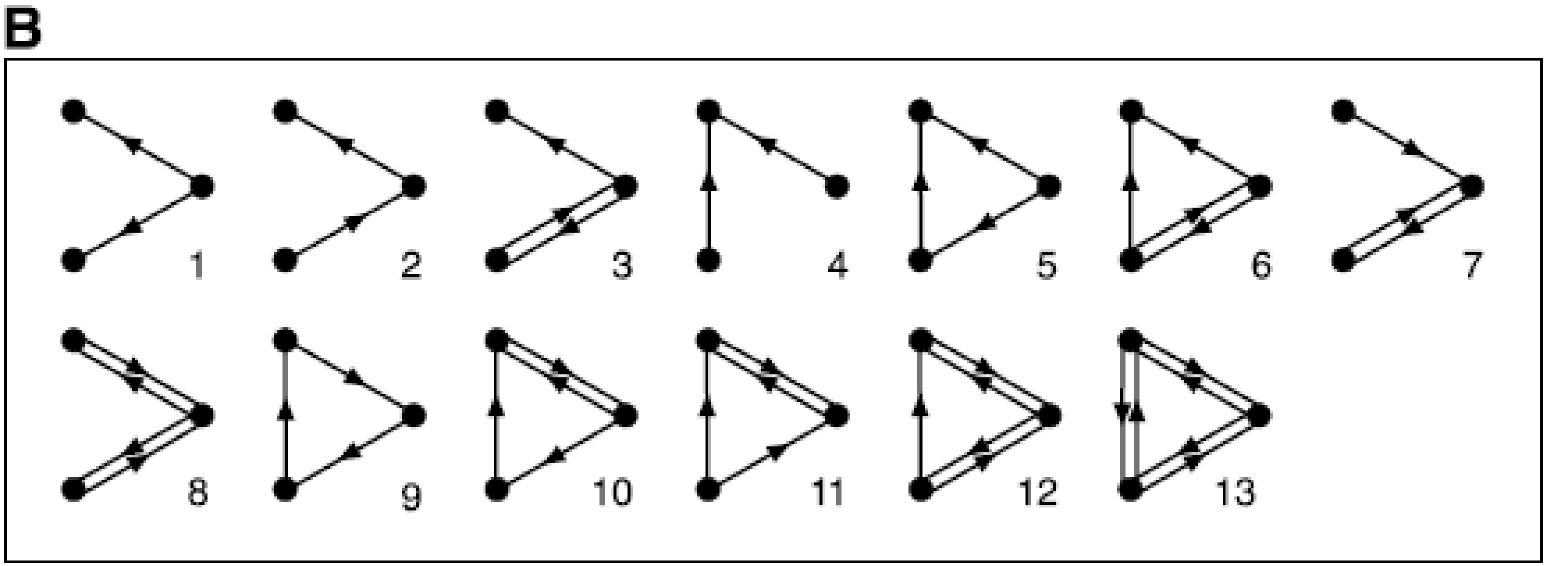


# Motifs

- Most networks have the same characteristics with respect to global measurements
  - can we say something about the local structure of the networks?
- Motifs: Find small subgraphs that are over-represented in the network

# Example

- Motifs of size 3 in a directed graph

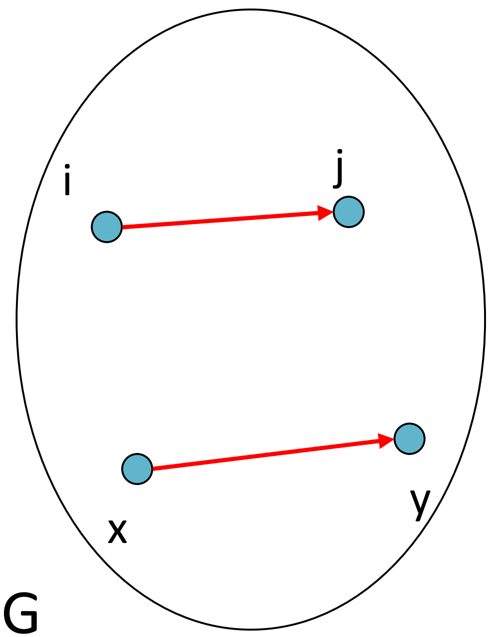


# Finding Interesting Motifs

- Sample a part of the graph of size **S**
- Count the **frequency of the motifs** of interest
- Compare against the **frequency of the motif in a random graph** with the same number of nodes and the same degree distribution

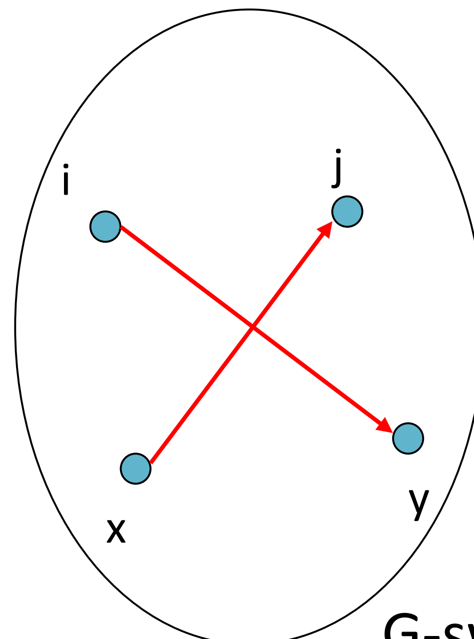
# Generating a Random Graph

- Find edges (i,j) and (x,y) such that edges (i,y) and (x,j) do not exist, and swap them
  - repeat for a large enough number of times




G

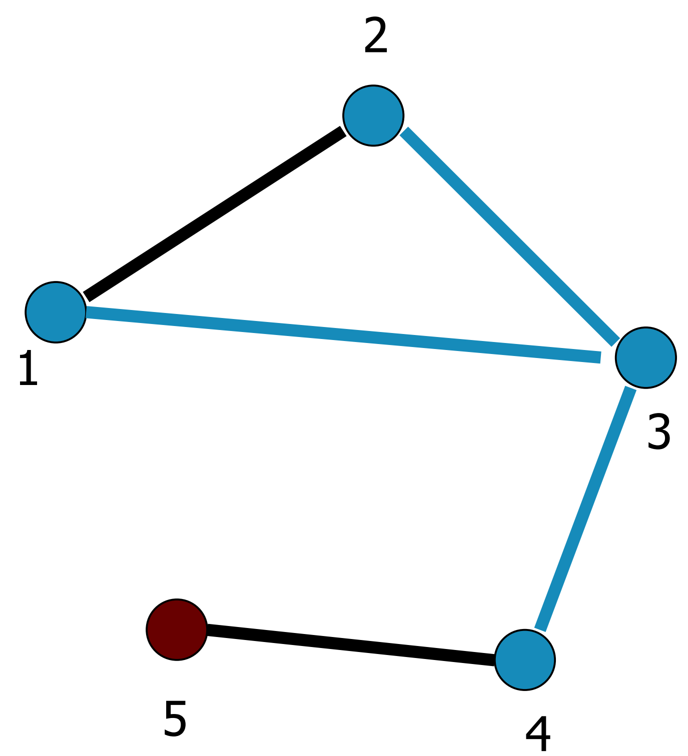
degrees of i,j,x,y
are preserved




G-swapped

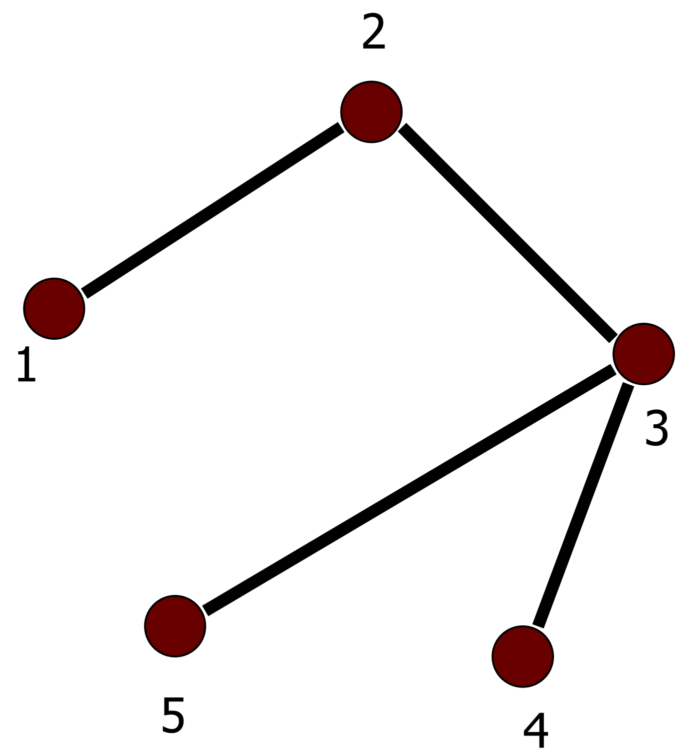
# Subgraphs

- Subgraph: Given $V' \subseteq V$, and $E' \subseteq E$, the graph $G'=(V',E')$ is a subgraph of G.
- Induced subgraph: Given $V' \subseteq V$, let $E' \subseteq E$ is the set of all edges between the nodes in V'. The graph $G'=(V',E')$, is an induced subgraph of G



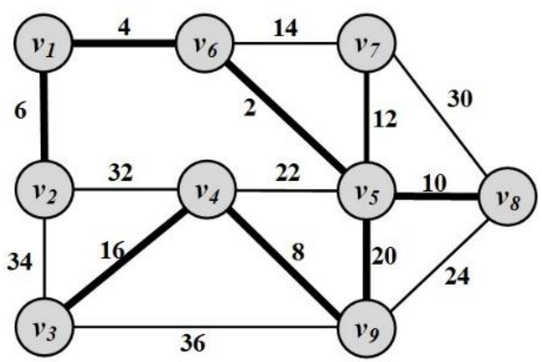
# Trees

- Connected Undirected graphs without cycles

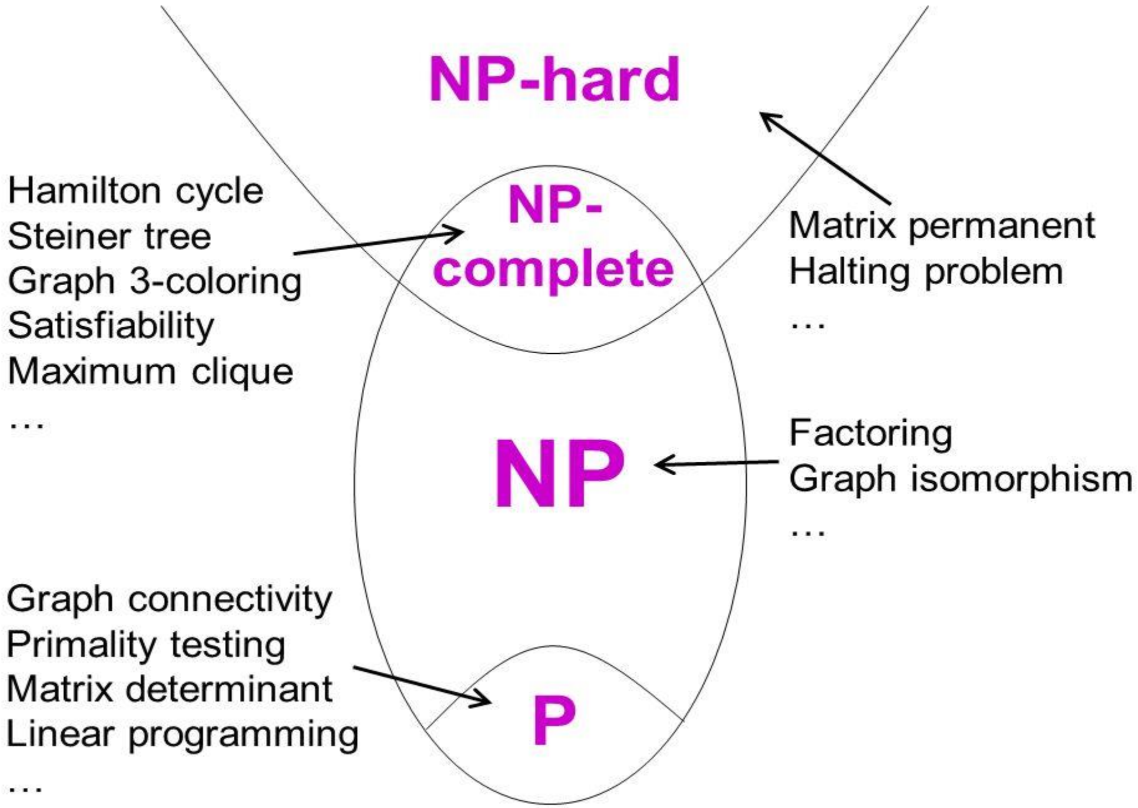


# Spanning Tree

- For any connected graph, the spanning tree is a subgraph and a tree that includes all the nodes of the graph
- There may exist multiple spanning trees for a graph
- The weigh of a spanning tree (among multiple spanning trees) of a graph is the summation of the edge weights in that spanning tree
- Minimum Spanning Tree (MST): The spanning tree with the minimum weight



# Classes of Complexity




**P:** Solvable in polynomial time

**NP:** Verified in polynomial time, but no known solution in polynomial time

**NP-hard:** At least as difficult as the hardest NP problems

**NP-complete**: The hardest of NP problems

# More Network Properties…

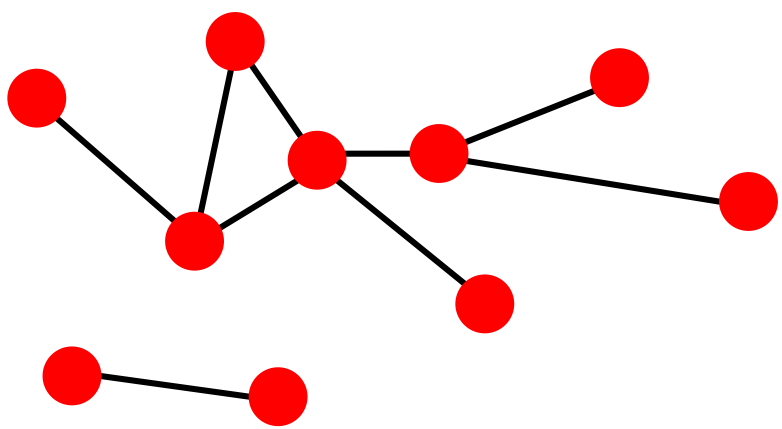
# Degree Distribution

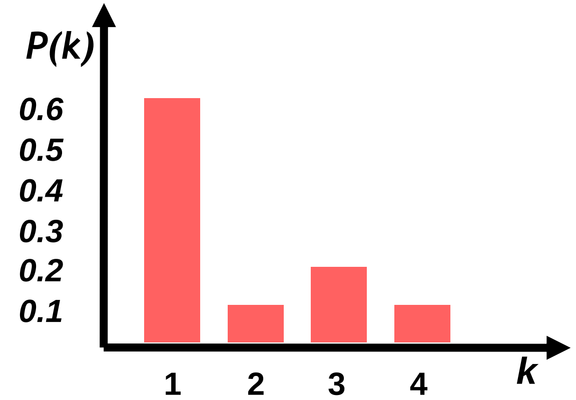
- **Degree distribution $P(k)$:** Probability that a randomly chosen node has degree $k$

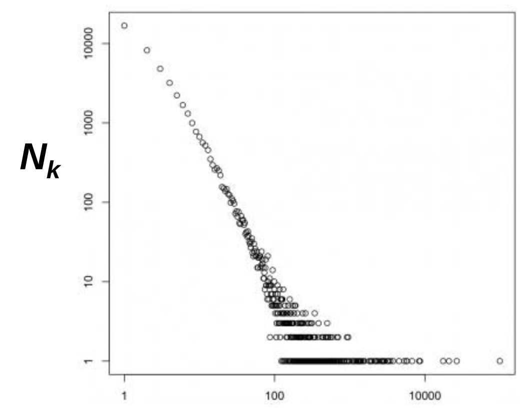
  $N_k$ = # nodes with degree $k$

- Normalized histogram:

  $P(k) = N_k / N$ ➔ plot





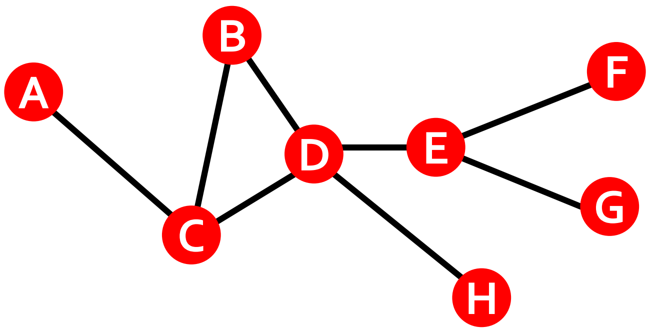



# Paths in a Graph

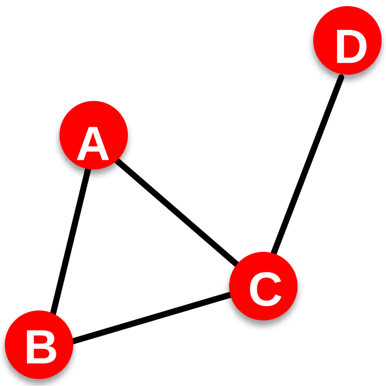
- A ***path*** is a sequence of nodes in which each node is linked to the next one

$$P_n = \{i_0, i_1, i_2, \ldots, i_n\} \qquad P_n = \{(i_0, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)\}$$

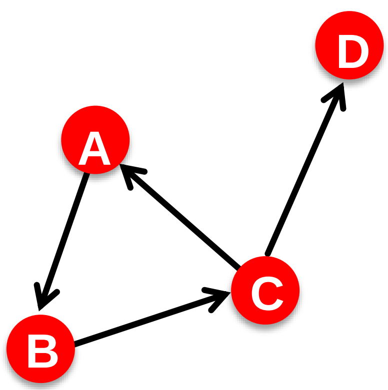
- Path can intersect itself and pass through the same edge multiple times
  - E.g.: ACBDCDEG
  - In a directed graph a path can only follow the direction of the "arrow"



# Distance in a Graph




$h_{B,D} = 2$

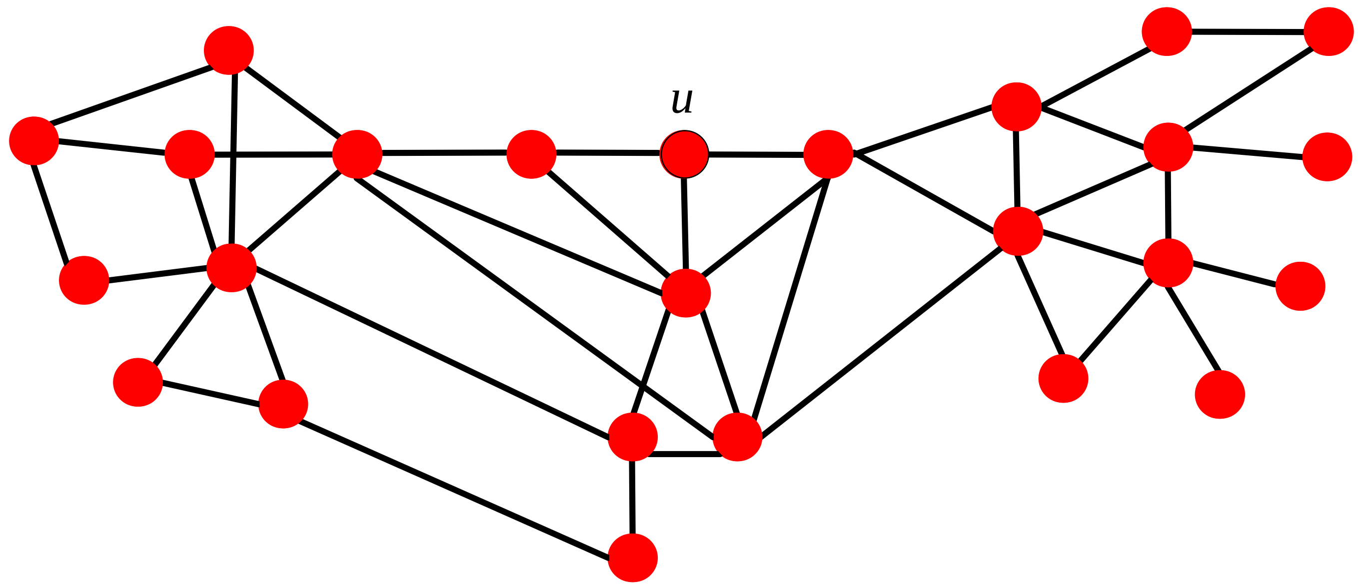



$h_{B,C} = 1,\ h_{C,B} = 2$

- **Distance (shortest path, geodesic)** between a pair of nodes is defined as the number of edges along the shortest path connecting the nodes
  - *If the two nodes are disconnected, the distance is usually defined as infinite
- In **directed graphs** paths need to follow the direction of the arrows
  - Consequence: Distance is **not symmetric:** $h_{A,C} \neq h_{C,A}$

# Finding Shortest Paths

- **Breadth First Search:**
  - Start with node $u$, mark it to be at distance $h_u(u)=0$, add $u$ to the queue
  - While the queue not empty:
    - Take node $v$ off the queue, put its unmarked neighbors $w$ into the queue and mark $h_u(w)=h_u(v)+1$



# Shortest Paths on Weighted Graphs

- Shortest paths on weighted graphs are harder to construct
  - There are several well known algorithms for finding single-source, or all-pairs shortest paths
- Single-source Shortest Path (SSSP)
  - Dijkstra’s algorithm (non-negative weights)
  - Bellman-Ford algorithm (allows negative weights)
- All-pairs Shortest Paths (APSP)
  - Floyd-Warshall algorithm (allows negative weights)
  - Johnson’s algorithm (allows negative weights)

# Network Diameter

- **Diameter:** the maximum (shortest path) distance between any pair of nodes in a graph

- **Average path length** for a connected graph (component) or a strongly connected (component of a) directed graph

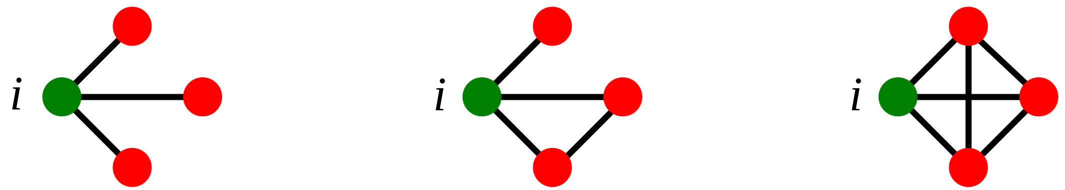
$$\bar{h} = \frac{1}{2E_{\max}} \sum_{i,j \neq i} h_{ij}$$

where $h_{ij}$ is the distance from node $i$ to node $j$

  - Many times we compute the average only over the connected pairs of nodes (that is, we ignore "infinite" length paths)

# Clustering Coefficient

- **Clustering coefficient:**
  - What portion of ***i***'s neighbors are connected?
  - Node ***i*** with degree $\boldsymbol{k_i}$
  - $C_i \in [0,1]$
  - $C_i = \dfrac{2e_i}{k_i(k_i - 1)}$ where $e_i$ is the number of edges between the neighbors of node $i$



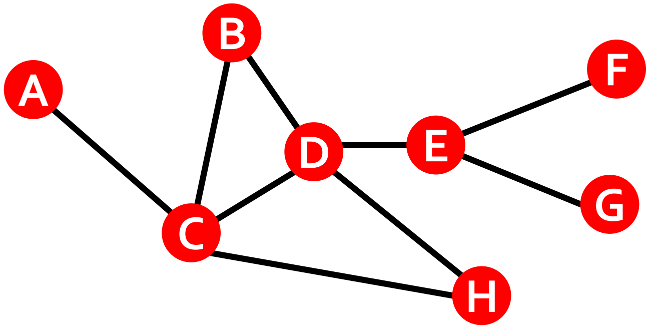

$C_i=0$ $C_i=1/3$ $C_i=1$

- **Average clustering coefficient:** $C = \dfrac{1}{N}\sum_{i}^{N} C_i$

# Clustering Coefficient: Example

- **Clustering coefficient:**
  - What portion of ***i***'s neighbors are connected?
  - Node ***i*** with degree $\boldsymbol{k_i}$
  - $C_i = \frac{2e_i}{k_i(k_i - 1)}$

where $e_i$ is the number of edges between the neighbors of node $i$




$k_B=2,\ \ e_B=1,\ \ \boldsymbol{C_B}=2/2 = 1$

$k_D=4,\ \ e_D=2,\ \ \boldsymbol{C_D}=4/12 = 1/3$

…

# Key Network Properties

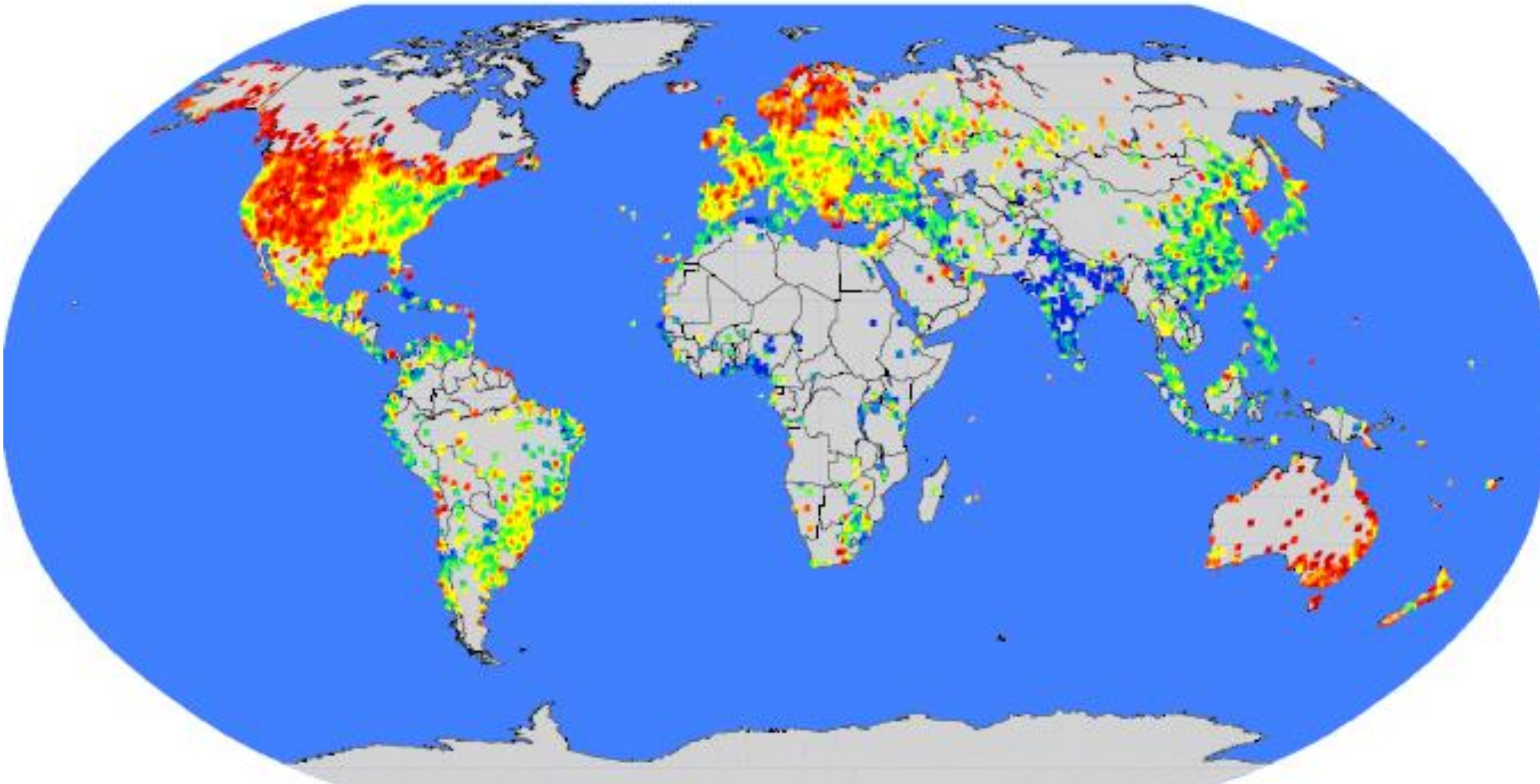
| | |
|---|---|
| **Degree distribution:** | $P(k)$ |
| **Path length:** | $h$ |
| **Clustering coefficient:** | $C$ |

# Let’s measure P(k), h and C on a real-world network!

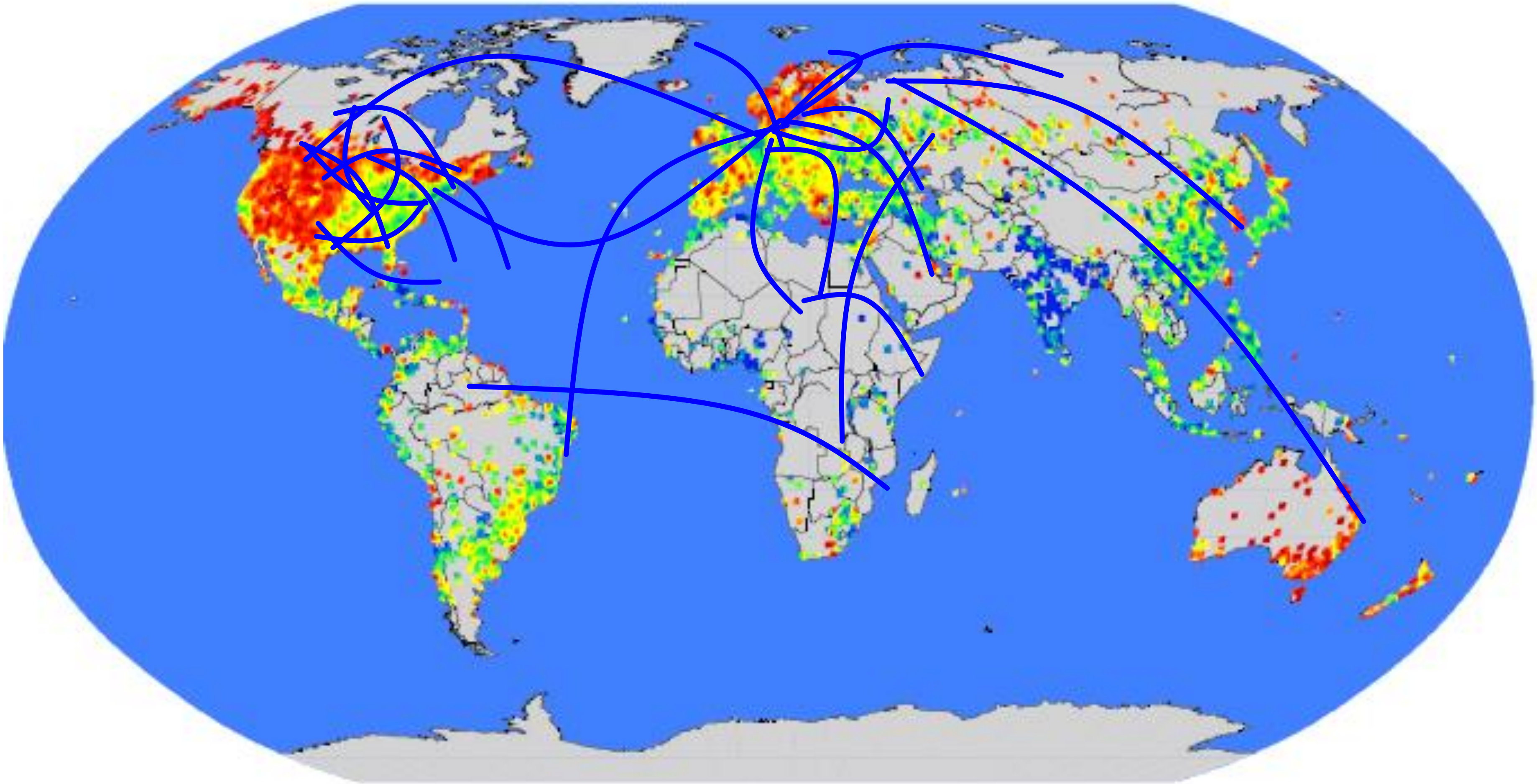
# The MSN Messenger


- **MSN Messenger activity in June 2006:**
  - 245 million users logged in
  - 180 million users engaged in conversations
  - More than 30 billion conversations
  - More than 255 billion exchanged messages

# Communication: Geography

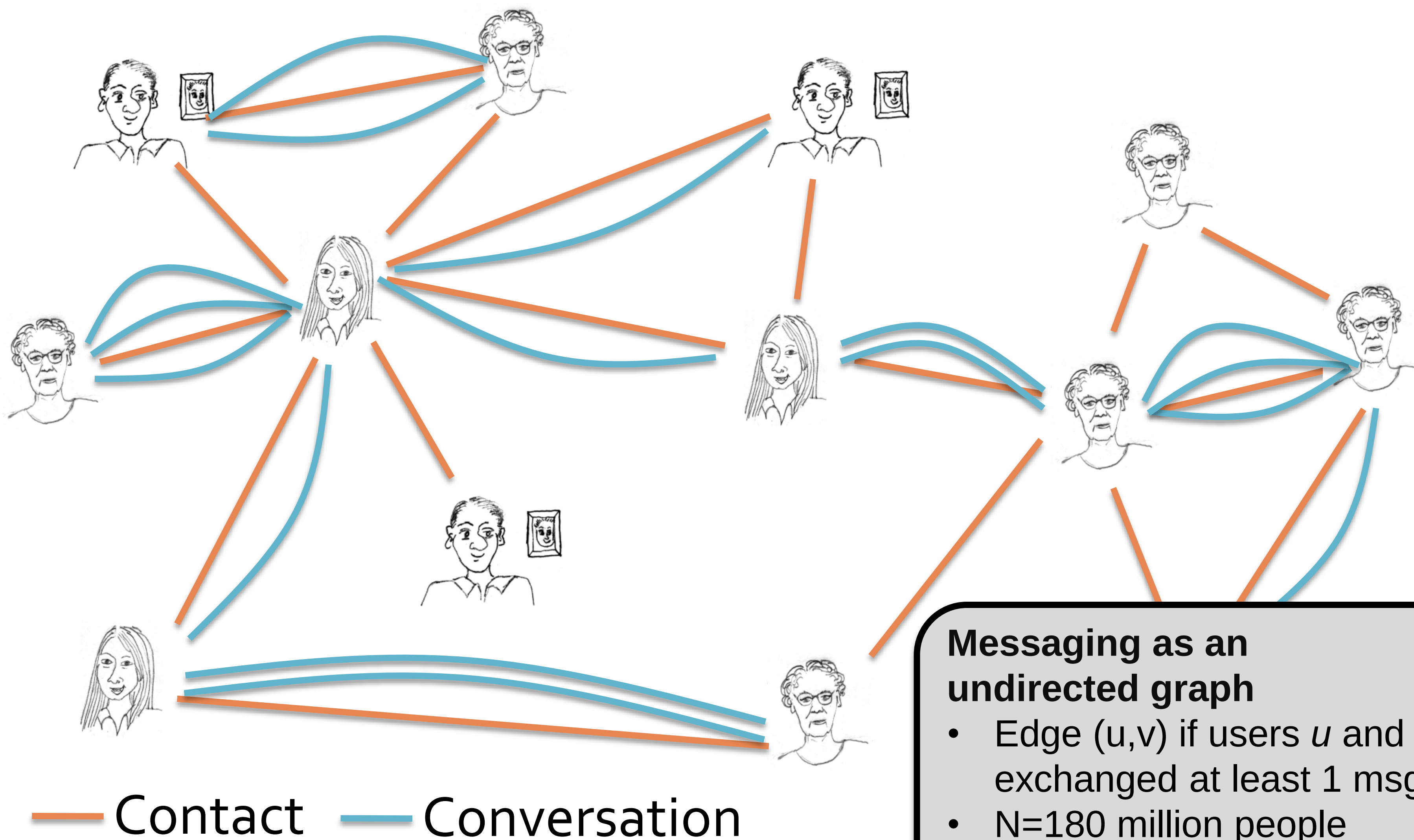


# Communication network



**Network:** 180M people, 1.3B edges

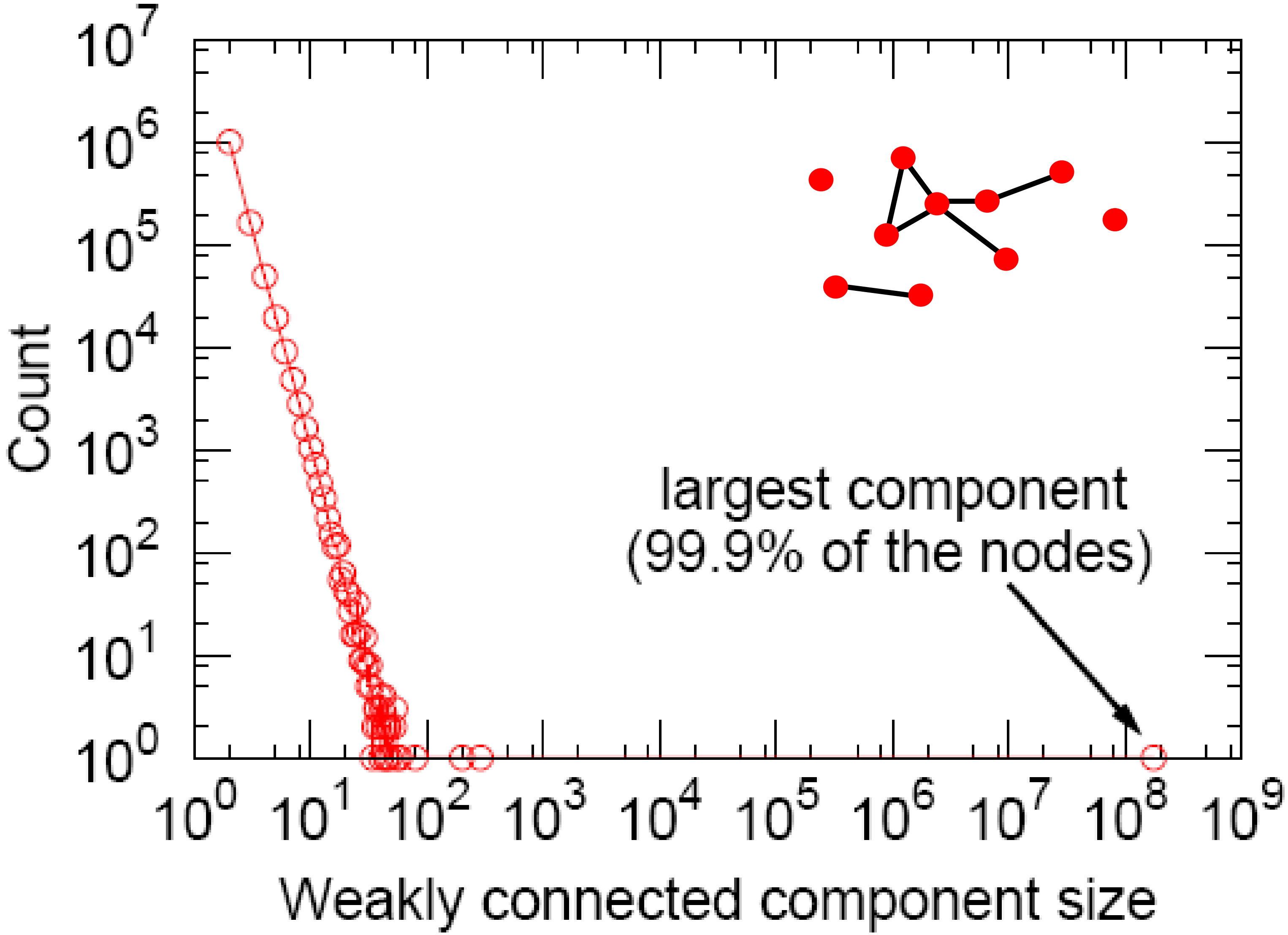
# Messaging as a Multigraph



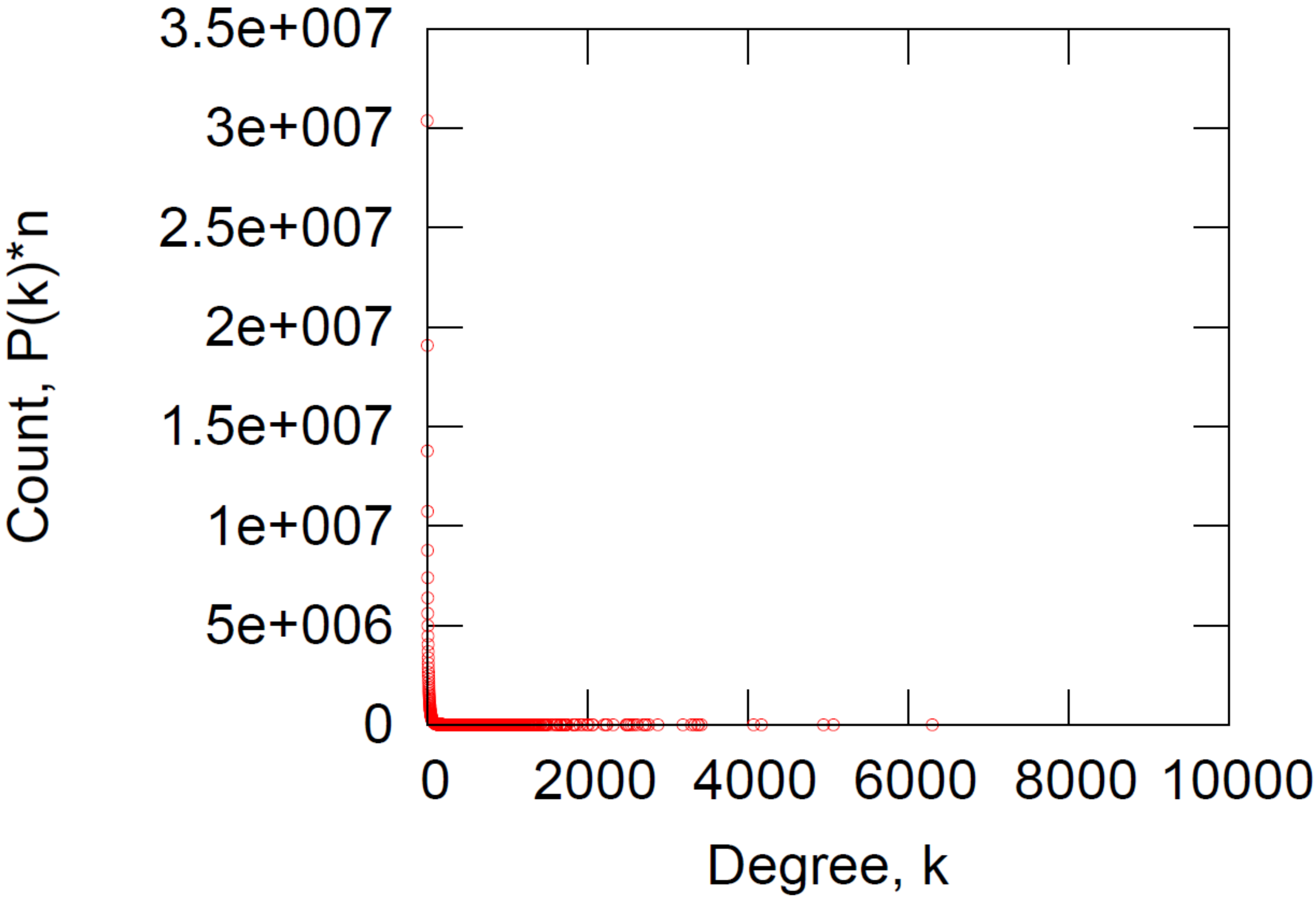

**Messaging as an undirected graph**

- Edge (u,v) if users *u* and *v* exchanged at least 1 msg
- N=180 million people
- E=1.3 billion edges

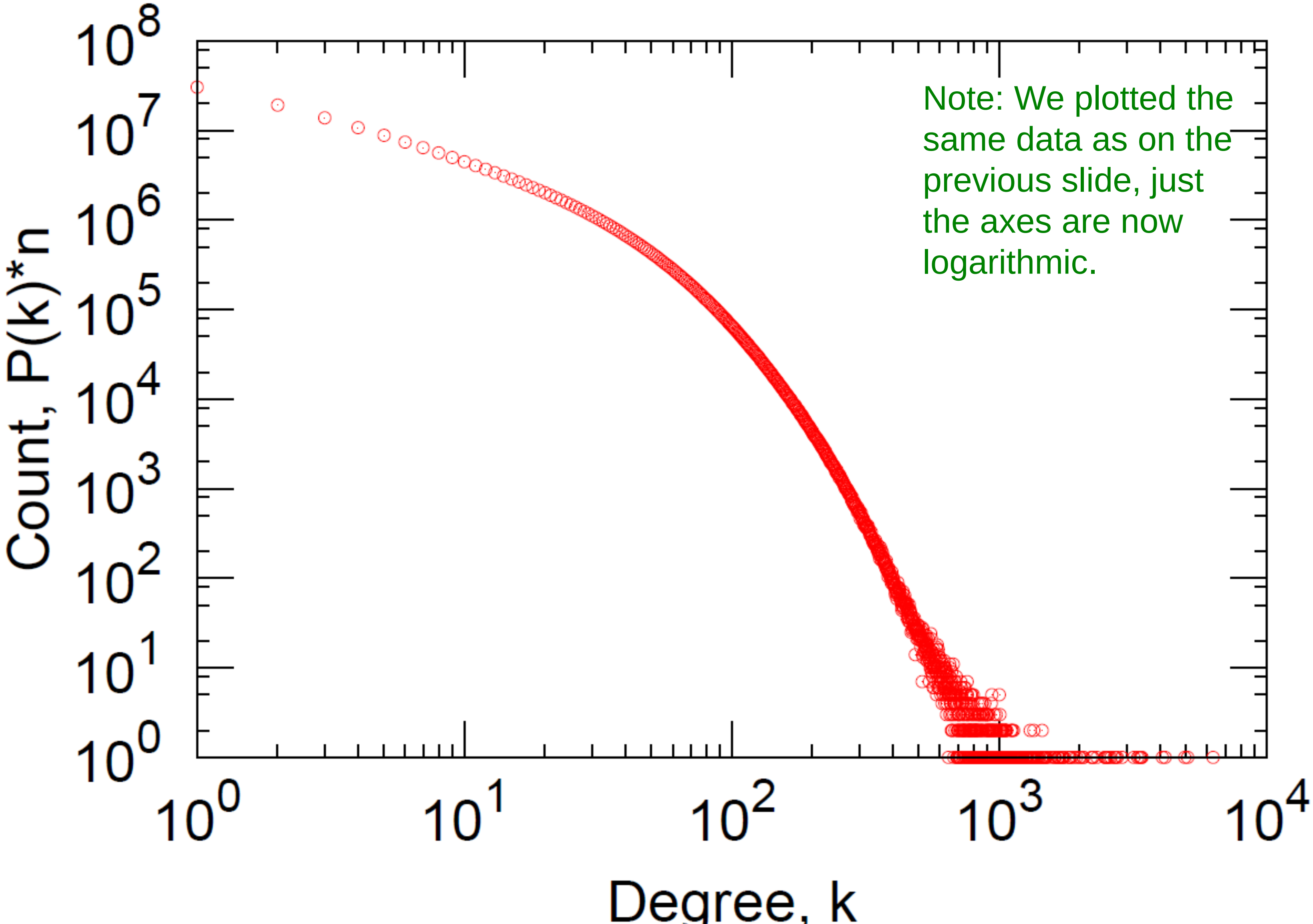
# MSN Network: Connectivity



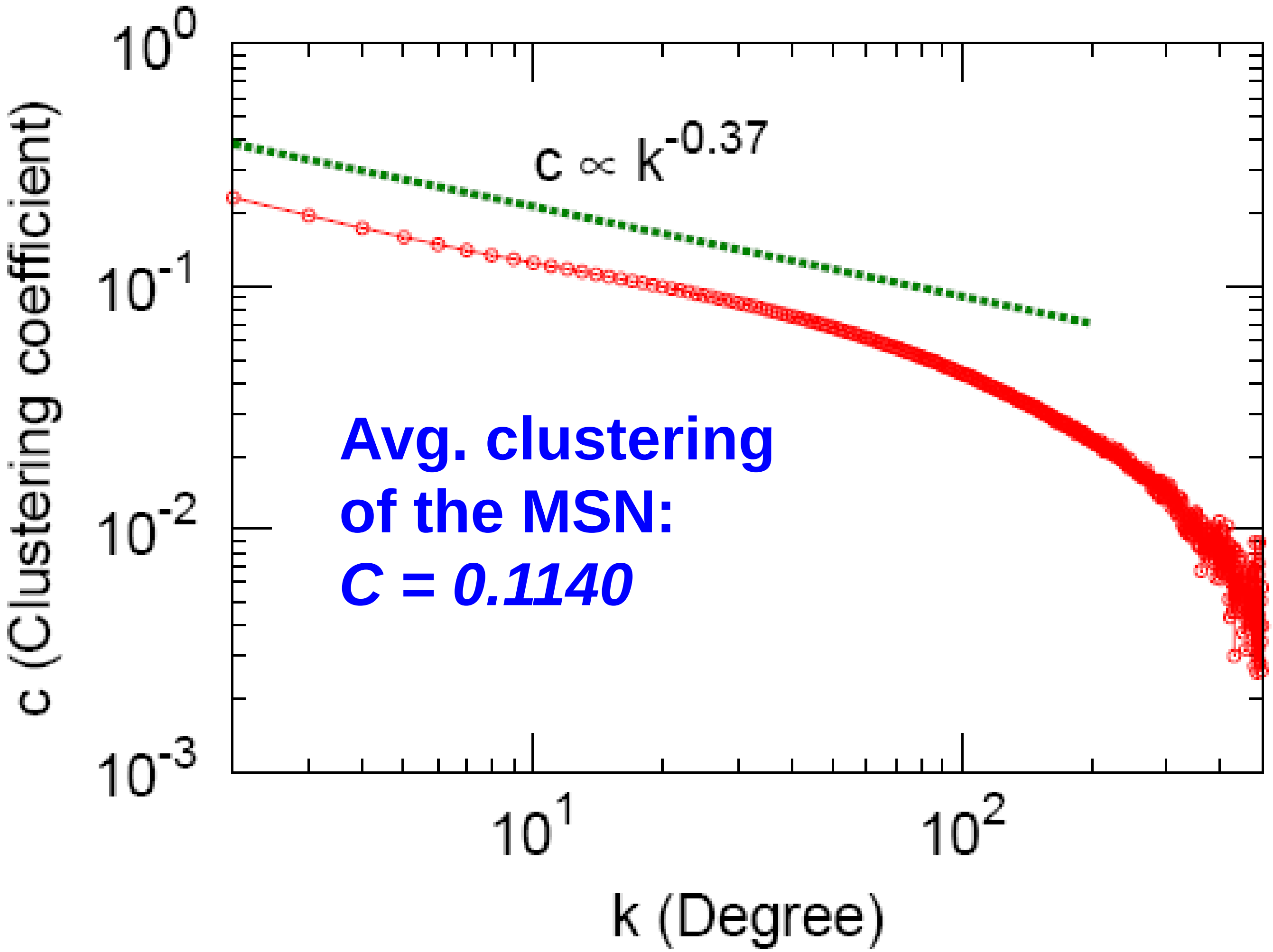
# MSN: Degree Distribution



# MSN: Log-Log Degree Distribution

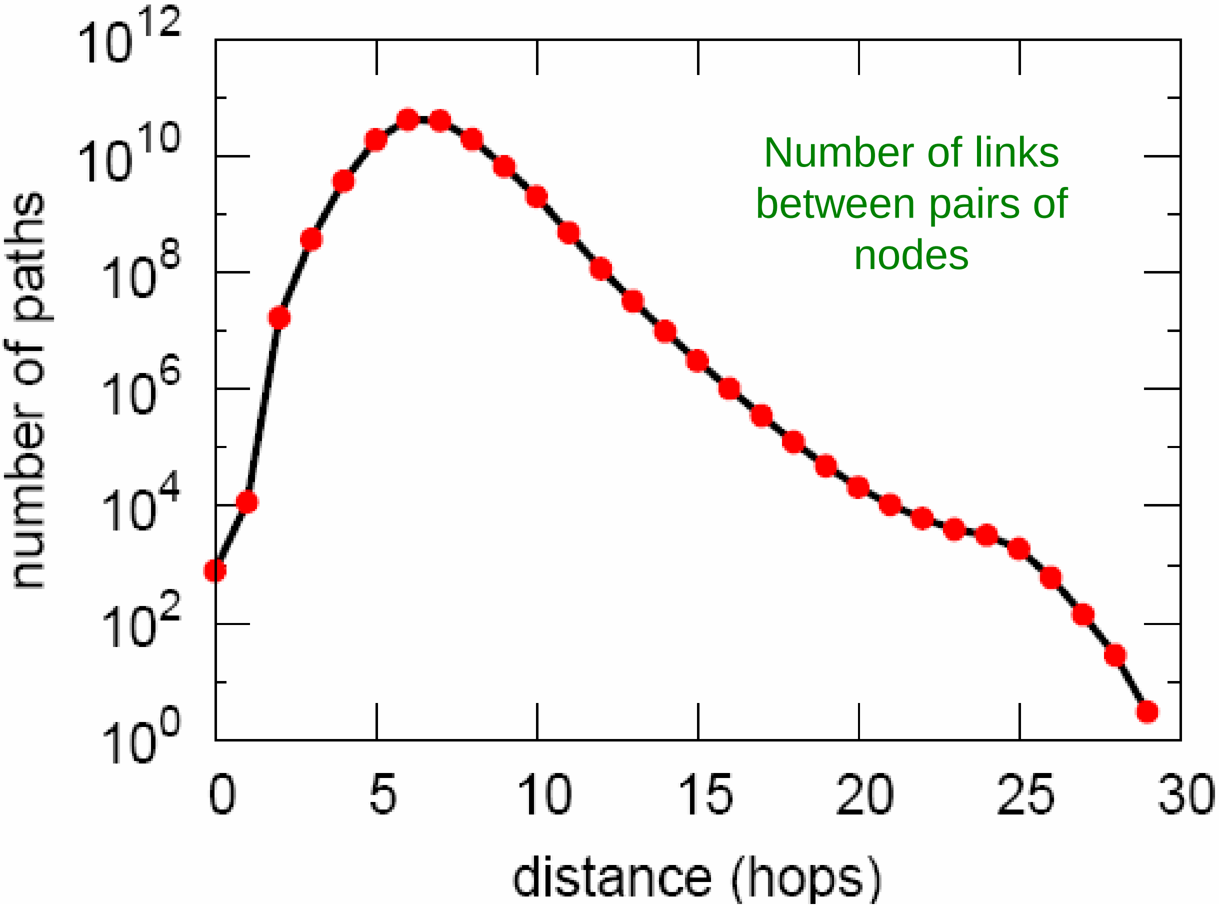


# MSN: Clustering




$C_k$: average $C_i$ of nodes $i$ of degree $k$: $C_k = \frac{1}{N_k} \sum_{i:k_i=k} C_i$

# MSN: Diameter




Avg. path length **6.6**

90% of the nodes can be reached in < 8 hops

# nodes as we do BFS out of a random node

| Steps | #Nodes |
|---|---|
| 0 | 1 |
| 1 | 10 |
| 2 | 78 |
| 3 | 3,96 |
| 4 | 8,648 |
| 5 | 3,299,252 |
| 6 | 28,395,849 |
| 7 | 79,059,497 |
| 8 | 52,995,778 |
| 9 | 10,321,008 |
| 10 | 1,955,007 |
| 11 | 518,410 |
| 12 | 149,945 |
| 13 | 44,616 |
| 14 | 13,740 |
| 15 | 4,476 |
| 16 | 1,542 |
| 17 | 536 |
| 18 | 167 |
| 19 | 71 |
| 20 | 29 |
| 21 | 16 |
| 22 | 10 |
| 23 | 3 |
| 24 | 2 |
| 25 | 3 |

# MSN: Key Network Properties

**Degree distribution:** *Heavily skewed*
*avg. degree= 14.4*

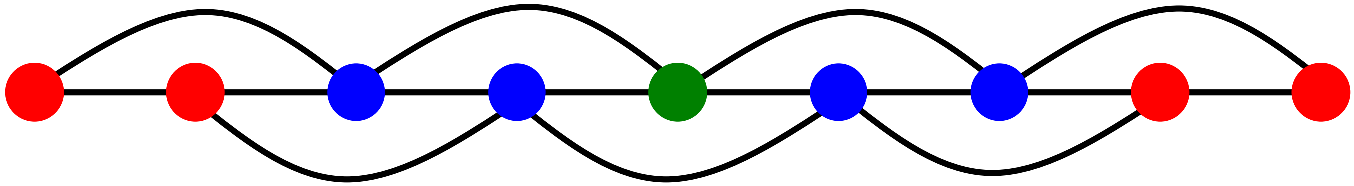
**Path length:** *6.6*

**Clustering coefficient:** *0.11*

**Are these values "expected"?**
**Are they "surprising"?**

**To answer this we need a null-model!**

# Is MSN Network like a "chain"?



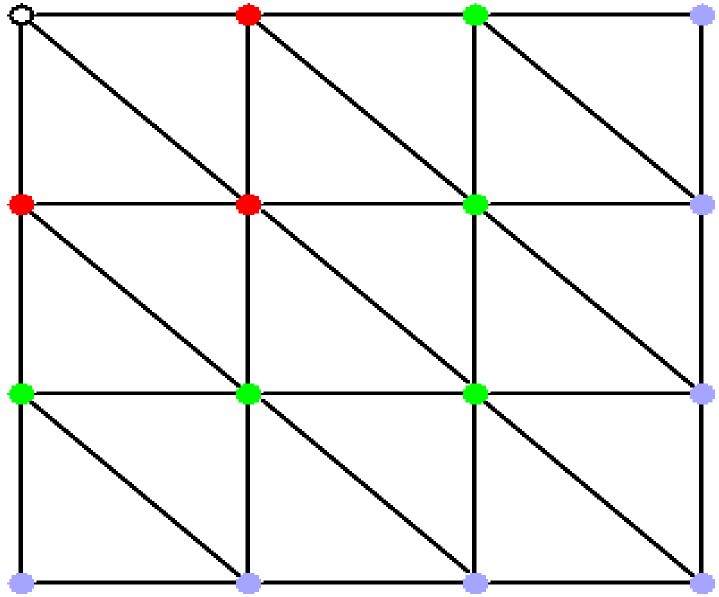
- $P(k) = \delta(k\text{-}4)$ $k_i = 4$ for all nodes
- $C = \frac{1}{N}\left(\frac{1}{2}(N-4) + 2 + 2\frac{2}{3}\right) = ½$ as $N \rightarrow \infty$
- Path length: $h_{max} = \frac{N-1}{2} = O(N)$
  - Avg. shortest path-length: $\bar{h} < \frac{2}{N(N-1)} \frac{N-1}{2} \frac{N(N-1)}{2} = O(N)$
- So, we have: **Constant degree,**

  **Constant avg. clustering coeff.**

  **Linear avg. path-length**

**Note about calculations:**
We are interested in quantities as graphs get large (N→∞)

# Is MSN Network like a "grid"?



- $P(k) = \delta(k-6)$
  - $k = 6$ for each inside node
- $C = 6/15$ for inside nodes
- **Path length:**

$$h_{\max} = O(\sqrt{N})$$

- **In general, for lattices:**
  - Average path-length is $\bar{h} \approx N^{1/D}$ (D… lattice dimensionality)
  - Constant degree, constant clustering coefficient

# What did we learn so far?

MSN Network is neither a chain nor a grid