York University
EECS 4101/5101, Winter 2023
Assignment 4

Due Date: March 19th, at 23:59

Dare to love yourself as if you were a rainbow with gold at both ends ...
Aberjhani

All problems are written problems; submit your solutions electronically **only via Crowdmark**. You are welcome to discuss the general idea of the problems with other students. However, you must write your answers individually and mention your peers (with whom you discussed the problems) in your solution. Please refer to the course webpage for guidelines on academic integrity.

Problem 1 Applications of Hashing [10 marks]

a) Given a set of n positive integers stored in an array A, describe an algorithm that runs in $O(n)$ and outputs any index i such that $A[i]$ and $A[i]^2$ are present in the set. If no such i exists, the algorithm must return -1. Assume indices start at 0. For example, for $A = [1, 8, 2, 3, 49, 2, 5, 7, 10]$, the output could be $i = 7$ (because $A[i] = 7$ and $A[i]^2 = 49$ is also present in the set). You may use a hash table and assume that uniform hashing assumption holds (that is, dictionary operations are supported in constant time).

You can write a pseudocode or describe your algorithm in English. Be succinct; you don't need more than a few sentences to describe a correct algorithm.

Answer: Perform a linear scan of A and store (using constant-time insertion) it in a hash table. This step takes $O(n)$. On the second linear scan, for each index i, check (in constant time search) if $A[i]^2$ belongs is present in the hash table.

b) Given a set of n integers stored in an array A, describe an algorithm that runs in $O(n)$ and outputs the length of the longest sub-array with a sum equal to 0. For example, for $A = [19, -8, -4, 6, -12, 5, 2, 3, 14, 27]$, the longest sub-array with elements summing up to 0 is $[-4, 6, -12, 5, 2, 3]$, and the output must be 6. You may use a hash table and assume that uniform hashing assumption holds (that is, dictionary operations are supported in constant time).

Hint: Form the "prefix-sum" array P, where $P[i]$ is the sum of the first $(i+1)$ indices in A. In the example above, $P = [19, 11, 7, 13, 1, 6, 8, 11, 25, 52]$. You can write a pseudocode or describe your algorithm in English. Be as succinct as you can.

Answer: The main observation is that, if $P[j] = P[i]$ (assume $j < i$) for some i and j, then the subarray $A[j+1], A[j+2], \ldots, A[i]$ has sum equal to 0. Therefore, the following approach works:

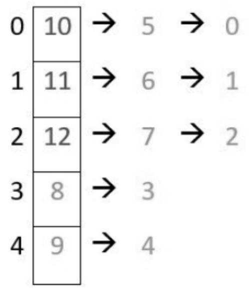
- From P, form P by a simple linear scan: set $P[1] = A[1]$ and $P[i] = P[i-1] + A[i]$. This takes $O(n)$.
- Let res indicate the output; intially $res = 0$. Perform a linear scan of P and store it in a hash table. For each index i, we use $P[i]$ as the key of the hash table and i as its value (so, we store them as a pair). For each index i, first, check if the key $P[i]$ belongs to the hash table. If not, we store $(P[i], i)$ in the hash table. If it does, we know $(P[i], j)$ is in the hash table (for some $j < i$). In this case, if $i - j > res$, we update $res = i - j$. All operations for index i take constant time; therefore, this step also takes $O(n)$.

Problem 2 Basics of Hashing [4+4 marks]

Assume that we have a hash table of size $M = 5$, we use the hash function $h(k) = k \bmod 5$, and we use chaining for collision resolution. Furthermore, assume that our universe is $U = \{0, 1, 2, \ldots, 12\}$.

a) Demonstrate the insertion of the keys $0, 1, 2, \cdots, 12$ into the (initially empty) hash table (in that order). You only need to draw the state of the hash table after all insertions are done.

Answer: This is how the hash table looks like (when, for example, we apply chaining).



b) **[bonus]** Suppose only two insertions of elements in U are made into the initially-empty hash table. If each pair of elements in U is equally likely to be inserted, calculate the probability that the second insertion caused a collision. In other words, given distinct k_1 and k_2 uniformly chosen from U, find that the probability that $h(k_1) = h(k_2)$.

Answer: There are $\binom{13}{2} = 78$ ways to select two items from the universe. The number of ways they can be equal is $3 \times \binom{3}{2} = 9$ (when they both map to indices 0, 1, or 2) plus $2 \times \binom{2}{2} = 2$ (when they both map to indices 3 or 4). So, the chance of two elements being equal is $\frac{9+2}{78}$.

Problem 3 Hash Functions [5 marks]

Assume a hash scheme in which keys are selected uniformly at random from the Universe set $U = \{1, 2, 3, \ldots, 600\}$. Consider the following two hash functions: $h_1(k) = k \ mod \ 6$ and $h_2(k) = 3k \ mod \ 6$. Which hash function is better? Justify your answer.

Answer: The first function is much better. For any number k, the value of $3k \ mode \ 6$ is either 0 or 3. So, this function maps all items in these two indices while other indices are left empty.

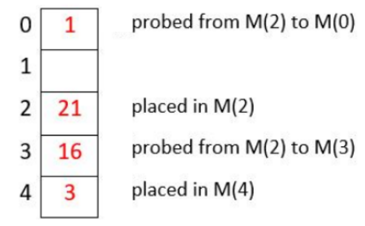
In general, a good hash function requires to depend on all parts of the input (that's why we want hash tables to have prime sizes; because division by a prim somehow makes all digits involved). In contrast, multiplying keys by 3 in the above example causes overlooking parts of the input.

Problem 4 Collision Handling [5+5=10 marks]

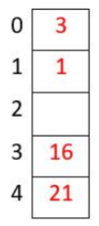
Consider a hash table dictionary with the universe $U = \{0, 1, 2, \ldots, 24\}$ and size $M = 5$. If items with keys $k = 21, 3, 16, 1$ are inserted in that order, draw the resulting hash table if we resolve collisions using:

a) Linear probing with $h(k) = (k + 1) \bmod 5$

b) Cuckoo hashing with $h_1(k) = k \bmod 5$ and $h_2(k) = \lfloor k/5 \rfloor$ **Answer:** see the figures below.



Linear probling



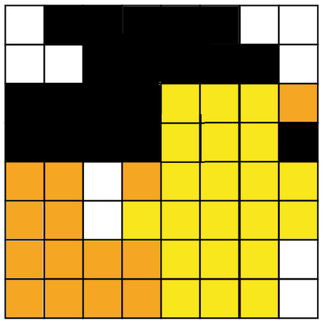
cuckoo hashing

For Cuckoo hashing, the following steps are taken:

- 21 is placed at $h_1(21) = 1$.
- 3 is placed at $h_1(3) = 3$.
- 16 is placed at $h_1(16) = 1$. So, 21 is kicked out from index 1 and is placed at $h_2(21) = 4$.
- 1 is placed at $h_1(1) = 1$. So, 16 is kicked out from index 1 and is placed at $h_2(16) = 3$. So, 3 is kicked out from index 3 and is placed at index $h_2(3) = 0$.

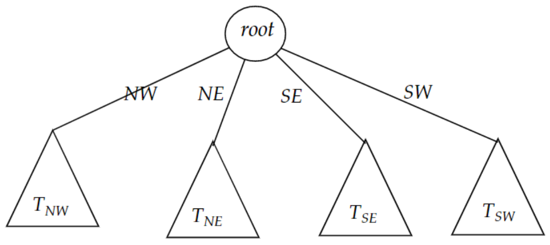
Problem 5 Image Encoding [5+5=10 marks]

Consider the following 8×8 pixel image of a (deformed) pokémon. Each pixel is either white (0), yellow (1), orange (2), or black (3). In this problem, we consider possible ways to encode the image.

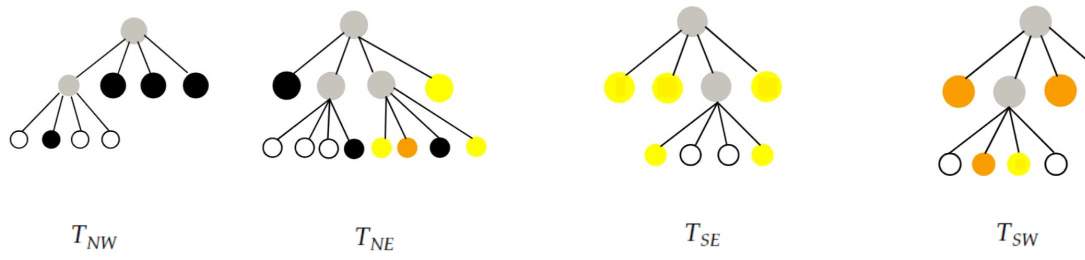


a) Consider a fixed-length encoding of the image as follows. We process the image row by row (from top to bottom), and for each row, we process pixels from left to right. For each pixel, we store 2 bits to specify its colour (00 for white, 01 for yellow, 10 for orange, and 11 for black). The first six bits will then be 001111.
Indicate what the length of the code will be. Show your work (you don't need to show the actual code).
Answer: There are $8 \times 8 = 64$ pixels, and 2 bits are used to encode the colour of each pixel. Therefore, the total length of the code will be $64 \times 2 = 128$.

b) One application of quadtrees is in image compression. An image is recursively divided into quadrants until the entire quadrant is only one colour. Using this rule, draw the quadtree of the pokémon image. Note that each tree leaf must store the colour of the area it represents. Draw a separate tree for each of $T_{NW}, T_{NE}, T_{SE}, T_{SW}$ in the following figure.



Answer: See the figures below:



Problem 6 Quad Trees [6 marks]

Let n be a large positive integer. Suppose we create n random numbers uniformly distributed in the range $[0, 1]$ and store them in an array $X = [x_1, x_2, \dots, x_n]$. Similarly, we store n uniformly-distributed random numbers in $[0..1]$ and store them in an array $Y = [y_1, y_2, \dots, y_n]$ and store another n uniformly-distributed random numbers (from the same range $[0..1]$) and store them in an array $Z = [z_1, z_2, \dots, z_n]$. Consider a set $P = \{p_1, p_2, \dots, p_n\}$ of n 3D points, where $p_i = \{x_i, y_i, z_i\}$. That is, P is a set of points distributed uniformly at random in a unit 3D box. We store P using a quadtree. Specify what the expected height of P is. You need to provide an exact formula and show your work.

Answer: Let $h(n)$ denote the expected height for the set of n numbers distributed uniformly in a cube. Note that the tree divides the cube into 8 sub-cubes. Since the points are distributed uniformly between these 8 sub-cubes, the expected number of points inside each sub-cube would be $n/8$. So, if $h(n)$ denote the expected height of the tree with n points, we can write $h(n) = 1 + h(n/8)$, $h(1) = 1$. To solve this recursion, we can write

$$\begin{aligned} h(n) &= 1 + h(n/8) \\ &= 2 + h(n/64) \\ &= \dots \\ &= i + h(n/8^i) \\ &= \log_8 n + h(1) \\ &= 1 + \log_8 n \end{aligned}$$

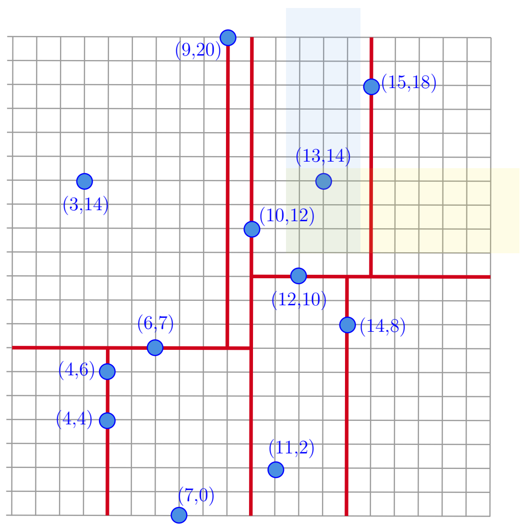
Therefore, the expected height of the tree is $\log_8 n + 1 = \Theta(\log n)$.

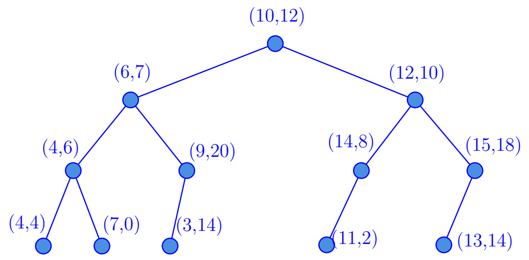
Problem 7 KD trees [4+4=8 marks]

Consider the following set of points in two dimensions:
$S = \{(13, 14), (11, 2), (4, 6), (12, 10), (9, 20), (10, 12), (3, 14), (14, 8), (15, 18), (4, 4), (6, 7), (7, 0)\}$.

a) Draw the analogous plane partition diagram and kd-tree.

Answer: See the figures below. The positions of $(4, 6)$ and $(4, 4)$ can be exchanged in the tree. Note that the next partition point is the one at index $\lceil n/2 \rceil$ from the sorted array (indices start at 0).





b) Show how a search for the points in the query rectangle $R = [11.5, 25] \times [14.5, 11]$ would proceed (that is, the box is formed by points (x, y) where $x \in [11.5, 25]$ and $y \in [11, 14.5]$). It suffices to show which nodes in the kd-tree are examined in the search.

Answer: The query nodes are as follows: $(10, 12)$, $(12, 10)$, $(15, 8)$, $(13, 14)$.