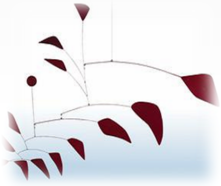
EECS 4101-5101
Advanced Data Structures



Shahin Kamali

Topic 4a - Biomial Heaps

York University

Picture is from the cover of the textbook CLRS.

Priority queues

- A **priority queue** is an abstract data type formed by a set S of key-value pairs
- **Basic operations** include:
  - **insert** (k) inserts a new element with key k into S
  - **get-Max** which returns the element of S with the largest key
  - **extract-Max** which returns the element of S with the largest key and delete it from S
- We are often given the whole data and need to **build** the data structure based on it.
  - Any data structure for a priority queue should be **constructed** efficiently.

Priority queue implementation

- What is a good implementation (data structure) for priority queues?

Priority queue implementation

- What is a good implementation (data structure) for priority queues?
- You have seen **binary heaps** before: get-Max runs in $O(1)$ and extract-Max and insert both take $\Theta(\log n)$ for n keys.

Priority queue implementation

- What is a good implementation (data structure) for priority queues?
- You have seen **binary heaps** before: get-Max runs in $O(1)$ and extract-Max and insert both take $\Theta(\log n)$ for n keys.
- Is a balanced binary search tree a good implementation of a priority queue?

Priority queue implementation

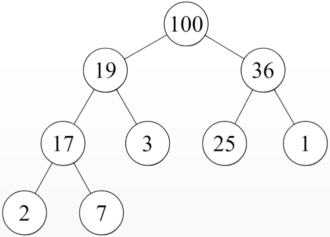
- What is a good implementation (data structure) for priority queues?
- You have seen **binary heaps** before: get-Max runs in $O(1)$ and extract-Max and insert both take $\Theta(\log n)$ for n keys.
- Is a balanced binary search tree a good implementation of a priority queue?
  - with a little augmentation, get-Max runs in $O(1)$ and extract-Max and insert both can run in $\Theta(\log n)$.

Priority queue implementation

- What is a good implementation (data structure) for priority queues?
- You have seen **binary heaps** before: get-Max runs in $O(1)$ and extract-Max and insert both take $\Theta(\log n)$ for n keys.
- Is a balanced binary search tree a good implementation of a priority queue?
  - with a little augmentation, get-Max runs in $O(1)$ and extract-Max and insert both can run in $\Theta(\log n)$.
- The problem with BSTs: it is costly to build them
  - How long does it take to form a BST from a given set of items?
  - It takes $\Omega(n \log n)$; otherwise you can sort them in $o(n \log n)$ by building the BST and doing an inoder traverse in $O(n)$.
  - We know we cannot comparison-sort in $o(n \log n)$ and hence cannot build the tree in such time.

Binary heaps

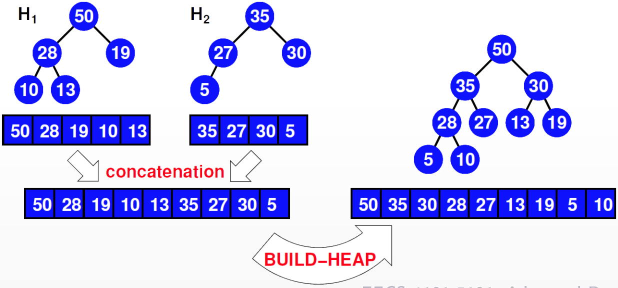
- A **heap** is a **tree** data structure
- For every node i other than the root, we have $key[parent[i]] \geq key[i]$.
- A **binary** heap is a complete binary tree which can be stored using an array.
  - build-heap takes $\Theta(n)$ time
  - insert, extract-Max take $\Theta(\log n)$
  - get-Max takes $O(1)$



100	*19*	*36*	*17*	*3*	*25*	*1*	*2*	*7*

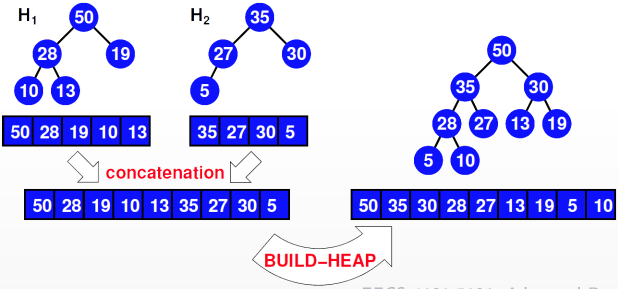
Binary heaps

- Suppose multiple priority queues on different servers.
- Occasionally a server must be rebooted, requiring two priority queues to be **merged**.
- With a typical binary heap, merging requires concatenating arrays and **re-running** build-heap; this takes $\Theta(n)$:'-(



Binary heaps

- Suppose multiple priority queues on different servers.
- Occasionally a server must be rebooted, requiring two priority queues to be **merged**.
- With a typical binary heap, merging requires concatenating arrays and **re-running** build-heap; this takes $\Theta(n)$:'-(
- When implementing an abstract data type always consider if you need it to be **mergable** or not.



Rethinking about Data Structure

- We would like to build a data structure for priority queues that:
  - supports insert, extract-Max, get-Max, and build efficiently (as in binary heaps)
  - merging two priority queues takes $o(n)$

Rethinking about Data Structure

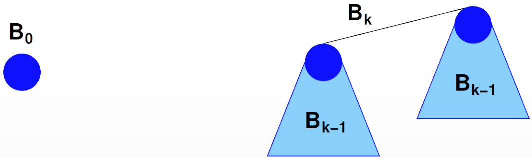
- We would like to build a data structure for priority queues that:
  - supports insert, extract-Max, get-Max, and build efficiently (as in binary heaps)
  - merging two priority queues takes $o(n)$
- Solution: **binomial heaps** which are mergable heaps that efficiently support
  - insert(H, x)
  - extract-Max(H)
  - get-Max(H)
  - build(A)
  - union(H_1, H_2) (merge)
  - increase-key(H, x, k)
  - delete(H, x)

Bionomial Trees

- A **binomial tree** is an ordered tree defined recursively
  - children of each node have a specific ordering (similar to 'left' and 'right' child in binary trees).

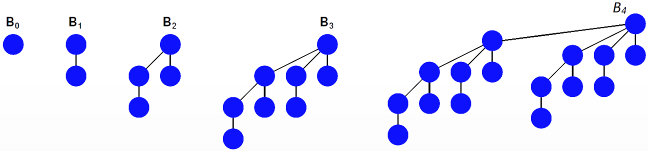
Bionomial Trees

- A **binomial tree** is an ordered tree defined recursively
  - children of each node have a specific ordering (similar to 'left' and 'right' child in binary trees).
- The base case for a binomial tree B_0 is a single node
- To build B_k, we take two copies of B_{k-1} and let the first child of the root of the second copy be the root of the first copy.



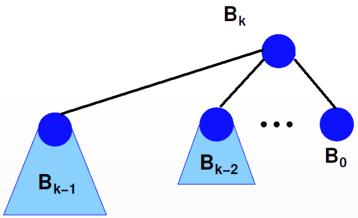
Binomial Trees

- A **binomial tree** is an ordered tree defined recursively
  - children of each node have a specific ordering (similar to 'left' and 'right' child in binary trees).
- The base case for a binomial tree B_0 is a single node
- To build B_k, we take two copies of B_{k-1} and let the first child of the root of the second copy be the root of the first copy.



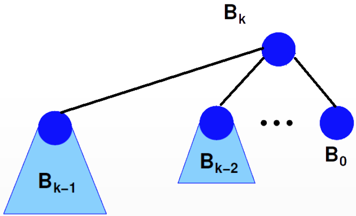
Fun with Binomial Trees

- Fun 1: The children of the root of the binomial tree B_k are the binomial trees $B_{k-1}, \ldots B_0$.



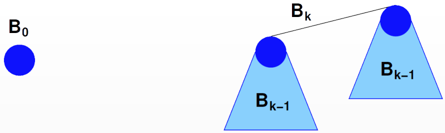
Fun with Binomial Trees

- Fun 1: The children of the root of the binomial tree B_k are the binomial trees $B_{k-1}, \dots B_0$.
  - Induction: assume it is true for all binomial trees B_i with $i \leq k-1$ (base easily holds).
  - The tree B_k has its first child as B_{k-1} (recursive construction).
  - With respect to other children, it is a binomial tree B_{k-1} and hence has children $B_{k-2}, \dots, B_0$ by induction hypothesis



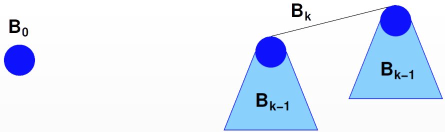
Fun with Bionomial Trees

- Fun 2: B_k has 2^k nodes:



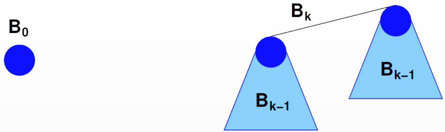
Fun with Bionomial Trees

- Fun 2: B_k has 2^k nodes:
  - The recursion is $N(B_k) = 2N(B_{k-1}), N(B_0) = 1$



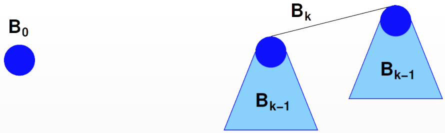
Fun with Bionomial Trees

- Fun 2: B_k has 2^k nodes:
  - The recursion is $N(B_k) = 2N(B_{k-1}), N(B_0) = 1$
- B_k has height k:



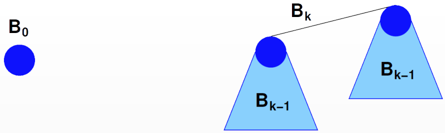
Fun with Bionomial Trees

- Fun 2: B_k has 2^k nodes:
  - The recursion is $N(B_k) = 2N(B_{k-1}), N(B_0) = 1$
- B_k has height k:
  - The recursion is $h(B_k) = h(B_{k-1}) + 1$:



Fun with Bionomial Trees

- Fun 2: B_k has 2^k nodes:
    - The recursion is $N(B_k) = 2N(B_{k-1}), N(B_0) = 1$
- B_k has height k:
    - The recursion is $h(B_k) = h(B_{k-1}) + 1$:
- Within B_k there are $\binom{k}{i}$ nodes at depth i.
    - The recursion is $ch(k, i) = ch(k-1, i-1) + ch(k-1, i)$
    - Solving this recursion gives $ch(k, i) = \binom{k}{i}$. To get an idea of the proof, note that $\binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i}$

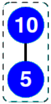


Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k

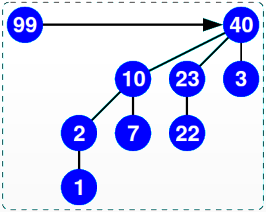


Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k



Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

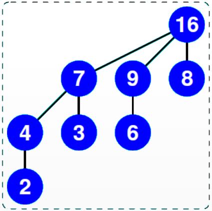
- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k

Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k

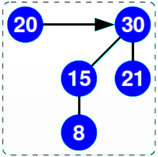


Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k

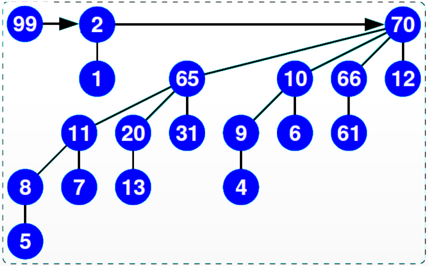


Binomial Heaps

Definition

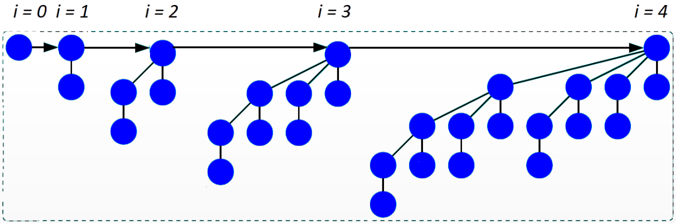
A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered ($key[parent[i]] \geq key[i]$)
- for each k there is at most one binomial tree of order k



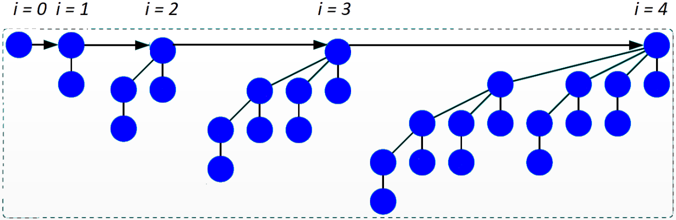
Number of Trees in Binomial Heaps

- How many trees are in a binomial heap of n nodes?

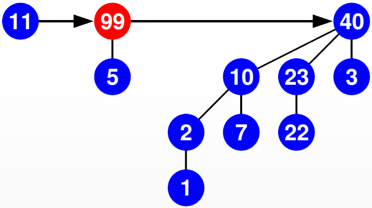


Number of Trees in Binomial Heaps

- How many trees are in a binomial heap of n nodes?
  - Let x be the number of trees
  - We express the number of nodes as a function of x
  - The number of trees is maximized when there is one tree of order i for any $i \in [0, x-1]$ (note that no two trees of same order can exist).
    - Recall that a binomial tree of order i has 2^i nodes.
    - We have $n = 1 + 2 + \ldots + 2^{x-1} = 2^x - 1$, i.e., $x = \lceil \log(n+1) \rceil$
- A binomial heap storing n keys has at most $\log(n+1)$ binomial trees.

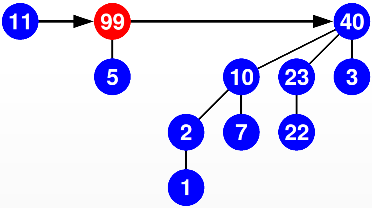


- For get-Max() operation, just follow the links connecting roots of binomial trees
  - The maximum element in all the heap is the max node, hence root, in one of the trees
  - E.g., max in the below heap is $\max\{11, 99, 40\} = 90$



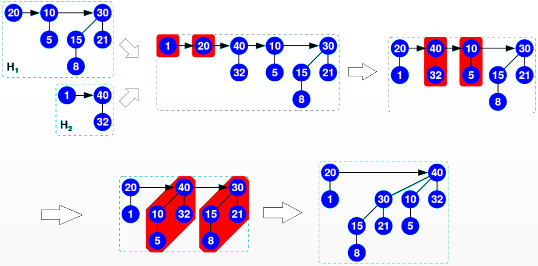
Finding Max in Binomial Heaps

- For get-Max() operation, just follow the links connecting roots of binomial trees
  - The maximum element in all the heap is the max node, hence root, in one of the trees
  - E.g., max in the below heap is $\max\{11, 99, 40\} = 90$
- There are $\log(n+1)$ trees and hence the time complexity is $\Theta(\log n)$.
  - It is a bit worse that $O(1)$ of get-Max() in binary heaps



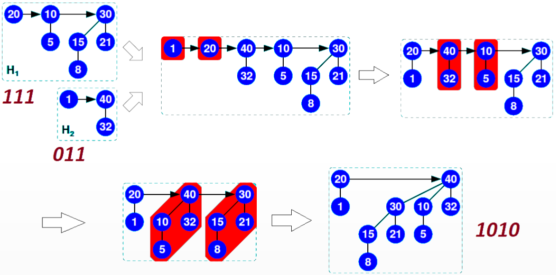
Merging of Two Binomial Heaps

- Union operation: we want to merge two heaps of sizes n_1 and n_2.
  - Similar to merge operation in merge sort, follow the links connecting roots of the heaps, and 'merge' them into one list (i.e., one heap).
  - If two trees of same order i are visited, merge them into a binomial tree of order $i + 1$
    - It is possible by the definition of binomial tree.
    - The tree with the smaller key in its root becomes a child of the other tree.
    - Two trees can be merged in $O(1)$.
  - When 3 trees of order i, merge the 2 older trees (keep the new one).



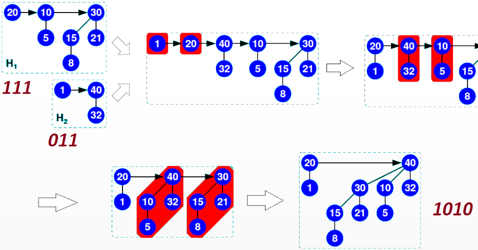
Merging of Two Binomial Heaps

- There is an analogy with **binary** addition: add bits and carry
  - Read from the least significant to the most significant bit (right to left)
  - 111 + 011 = 1010; "1010" means 1 tree of order 3, 0 tree of order 2, 1 tree of order 1, and 0 tree of order 0.



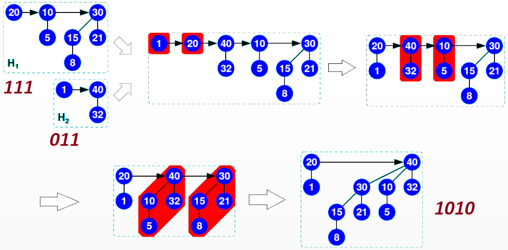
Merge Time Complexity

- What is time complexity of merge?
  - Each merge operation takes $O(1)$.
  - For each tree rank, there will be at most one merge
  - The total time complexity is $O(\log(n_1) + \log(n_2)) = O(2\log(\max\{n_1, n_2\})) = O(\log n)$ where n is the size after the merge



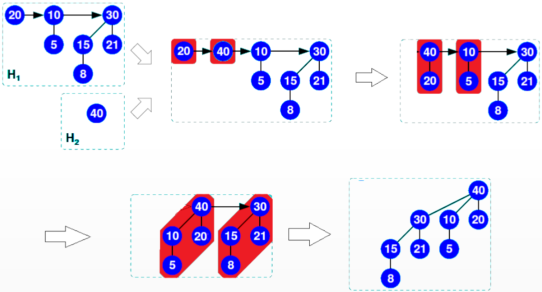
Merge Time Complexity

- What is time complexity of merge?
    - Each merge operation takes $O(1)$.
    - For each tree rank, there will be at most one merge
    - The total time complexity is $O(\log(n_1) + \log(n_2)) = O(2\log(\max\{n_1, n_2\})) = O(\log n)$ where n is the size after the merge
- **It is possible to merge two binomial heaps in $O(\log n)$ where n is the number of keys after the merge.**



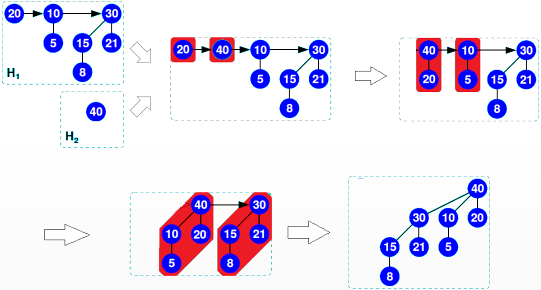
Insert Operation

- To insert a new key x to the priority queue:
  - Create a new binomial heap of size 1 (order 0) with the new key
  - Return the union of the old heap with the new one (e.g., Insert(40))



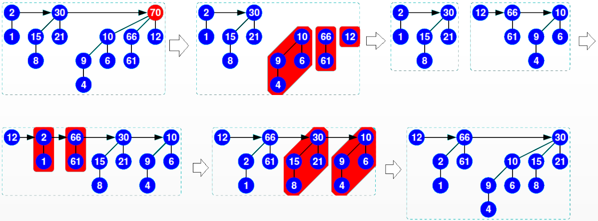
Insert Operation

- To insert a new key x to the priority queue:
    - Create a new binomial heap of size 1 (order 0) with the new key
    - Return the union of the old heap with the new one (e.g., Insert(40))
    - The time complexity is similar to merge.
- **It is possible to insert a new item to a binomial heap in** $O(\log n)$, which is as good as binary heaps



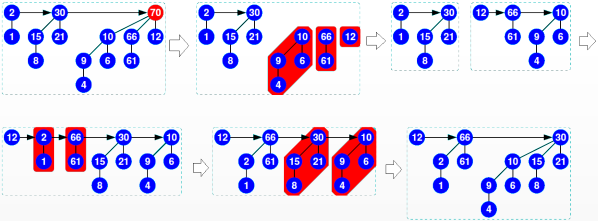
Extract-Max Operation

- To extract max, first search and find the maximum.
  - Assuming max is in a binomial tree of order k, its children are k binomial trees of order $0, 1, 2, \ldots, k-1$
  - Delete max and create a new binomial heap formed by these trees.
  - Merge the old heap and the new one.
  - The time complexity is $O(\log n)$ for finding the max and $O(\log n)$ for merging the two heaps, i.e., $O(\log n)$ in total



Extract-Max Operation

- To extract max, first search and find the maximum.
  - Assuming max is in a binomial tree of order k, its children are k binomial trees of order $0, 1, 2, \ldots, k-1$
  - Delete max and create a new binomial heap formed by these trees.
  - Merge the old heap and the new one.
  - The time complexity is $O(\log n)$ for finding the max and $O(\log n)$ for merging the two heaps, i.e., $O(\log n)$ in total
- **It is possible to extract maximum element in a binomial heap in** $O(\log n)$, which is as good as binary heaps



Bionmial Heaps Review

- Get-Max can be done in $\Theta(\log n)$ (a bit slower than $\Theta(1)$ of binary heaps).
- Merge can be done in $\Theta(\log n)$ (much better than $\Theta(n)$ of binary heaps).
- Insert and Extract-Max can be done in $\Theta(\log n)$ (similar to binary heaps)

Increase Key

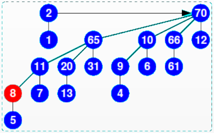
- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.

Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
  - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)

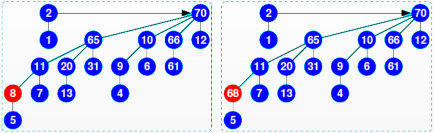
Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
  - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)
- Increase the key and 'float' it upward until $key[parent[i]] \geq key[i]$ (e.g., increase '8' to '68').



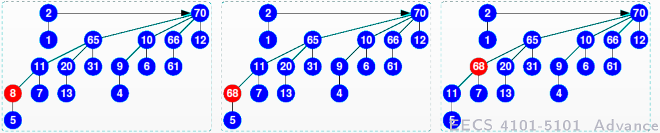
Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
  - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)
- Increase the key and 'float' it upward until $key[parent[i]] \geq key[i]$ (e.g., increase '8' to '68').



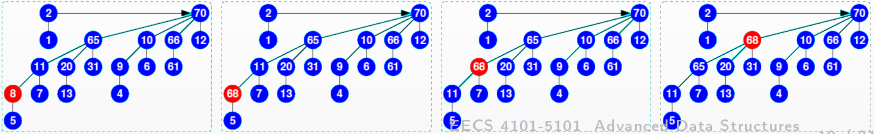
Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
  - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)
- Increase the key and 'float' it upward until $key[parent[i]] \geq key[i]$ (e.g., increase '8' to '68').



Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
    - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)
- Increase the key and 'float' it upward until $key[parent[i]] \geq key[i]$ (e.g., increase '8' to '68').



Increase Key

- Increase(a,x): assume you are given a pointer to a key a and want to increase it by x.
  - Note that if the pointer is not given, you need to search for the key, which takes $\Theta(n)$ in any heap (heaps are NOT good for searching)
- Increase the key and 'float' it upward until $key[parent[i]] \geq key[i]$ (e.g., increase '8' to '68').
- Time is proportional to the height of a binomial tree, i.e., the order of the tree
  - Recall that a binomial tree of order k has 2^k nodes, so, the order and hence the height of any tree in the heap is $O(\log n)$.
- **Increase the key of a given node can be done in time $\Theta(\log n)$.**

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it
  - Call Increase-key to set the key to ∞.
  - Call Extract-Max to remove the largest item; this would remove our node from the heap
- Time is $O(\log n)$ for Increase-key and $O(\log n)$ for Extract-Max.

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it
  - Call Increase-key to set the key to ∞.
  - Call Extract-Max to remove the largest item; this would remove our node from the heap
- Time is $O(\log n)$ for Increase-key and $O(\log n)$ for Extract-Max.

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it
  - Call Increase-key to set the key to ∞.
  - Call Extract-Max to remove the largest item; this would remove our node from the heap
- Time is $O(\log n)$ for Increase-key and $O(\log n)$ for Extract-Max.

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it
    - Call Increase-key to set the key to ∞.
    - Call Extract-Max to remove the largest item; this would remove our node from the heap
- Time is $O(\log n)$ for Increase-key and $O(\log n)$ for Extract-Max.

Delete

- Delete(a): assume you are given a pointer to a key a and want to delete it
  - Call Increase-key to set the key to ∞.
  - Call Extract-Max to remove the largest item; this would remove our node from the heap
- Time is $O(\log n)$ for Increase-key and $O(\log n)$ for Extract-Max.
- **Deleting a given node can be done in time $O(\log n)$.**

Binomial Heaps Summary

- Given a key (a pointer to its node), we can increase or delete that node in $O(\log n)$.

> **Theorem**
>
> *Priority queries can be implemented with binomial tree so that* ***Get-Max***, ***Merge***, ***Extract-Max***, ***Increase*** *(with given pointer) and* ***delete*** *(with given pointer) can all be performed in* $O(\log n)$.