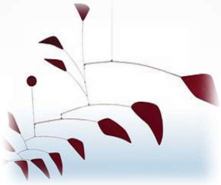


EECS 4101-5101
Advanced Data Structures

Shahin Kamali

Topic 2a AVL Trees and Augmentation

York University

Picture is from the cover of the textbook CLRS.

Dictionary ADT

Definition

A *dictionary* is a collection S of *items*, each of which contains a **key** and some **data**, and is called a **key-value pair** (KVP).

- It is also called an **associative array**, a **map**, or a **symbol table**.
- Keys can be compared and are (typically) unique.
- We often focus on keys; associating data with keys is easy.

Dictionary ADT

Definition

A *dictionary* is a collection S of *items*, each of which contains a **key** and some **data**, and is called a **key-value pair** (KVP).

- It is also called an **associative array**, a **map**, or a **symbol table**.
- Keys can be compared and are (typically) unique.
- We often focus on keys; associating data with keys is easy.

Main Operations:

- *search*(x): return true iff $x \in S$
- *insert*(x, v): $S \leftarrow S \bigcup \{x\}$
- *delete*(x): $S \leftarrow S / \{x\}$
- additional: *join*, *isEmpty*, *size*, **etc.**

Examples: student database, symbol table, license plate database

Optional Operations

- In addition to the main operations (search, insert, delete), the followings are useful:
  - *predecessor*(x): return the largest $y \in S$ such that $y < x$
  - *successor(x)*: return the smallest $y \in S$ such that $y > x$
  - *rank*(x) : return the index of x in the sorted array
  - *select*(i): return the key at index i in the sorted array $\rightarrow$ i'th order statistic
  - *isEmpty*(x): return true if S is empty

Dictionaries

- Is dictionary an abstract data type or a data structure?

Dictionaries

- Is dictionary an abstract data type or a data structure?
  - It is an abstract data type; we did not discuss implementation.
  - Different data structures can be used to implement dictionaries.

Elementary Implementations

- Common assumptions:
  - Dictionary has n KVPs
  - Each KVP uses constant space (if not, the "value" could be a pointer)
  - Comparing keys takes constant time
- **Unsorted array or linked list**

  search $\Theta(n)$
  insert $\Theta(1)$
  delete $\Theta(n)$ (need to search)
- **Sorted array**

  search $\Theta(\log n)$
  insert $\Theta(n)$
  delete $\Theta(n)$

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array				
sorted linked-list				
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list				
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$			
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$		
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST				
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$			
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$		
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST				
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$			
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$		
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables				
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$			
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$		
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	$\Theta(n+a)$
skip list				

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	$\Theta(n+a)$
skip list	$\Theta(n)^*$			

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	$\Theta(n+a)$
skip list	$\Theta(n)^*$	$\Theta(\log n)^*$		

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	$\Theta(n+a)$
skip list	$\Theta(n)^*$	$\Theta(\log n)^*$	$\Theta(\log n)^*$	

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Data Structures for Dictionaries

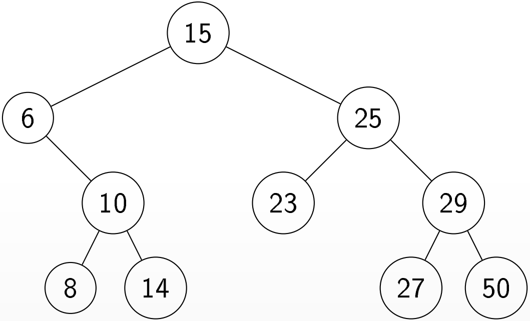
	space	search	insert/delete	predecessor
unsorted array,linked list	$\Theta(n+a)$	$\Theta(n)$	$\Theta(1)/\Theta(n)$	$\Theta(n)$
sorted array	$\Theta(n+a)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$
sorted linked-list	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
unbalanced BST	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
balanced BST	$\Theta(n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
hash tables	$\Theta(n+a)$	$\Theta(1)^*$	$\Theta(1)^*$	$\Theta(n+a)$
skip list	$\Theta(n)^*$	$\Theta(\log n)^*$	$\Theta(\log n)^*$	$\Theta(\log n)^*$

- n: number of KVPs.
- a: the length of array; when we use sorted/unsorted arrays, $a \geq n$.
- $*$: expected time/space

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

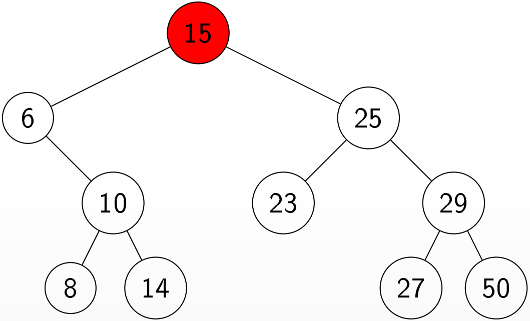
Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

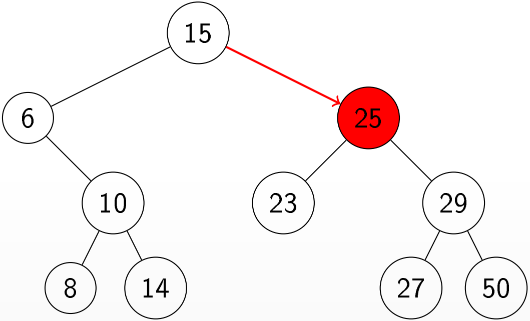
Example: *search*(24)



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

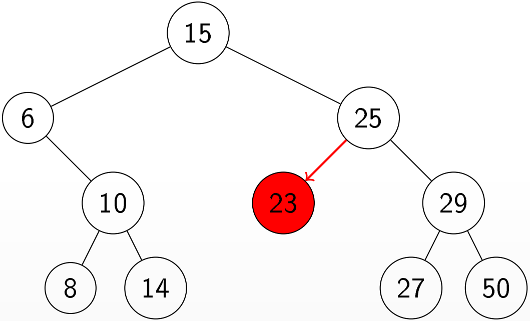
Example: *search*(24)



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

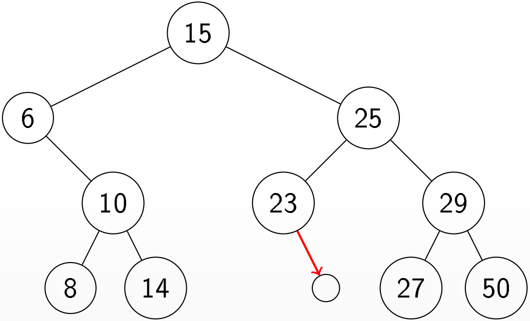
Example: *search*(24)



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

Example: *search*(24)

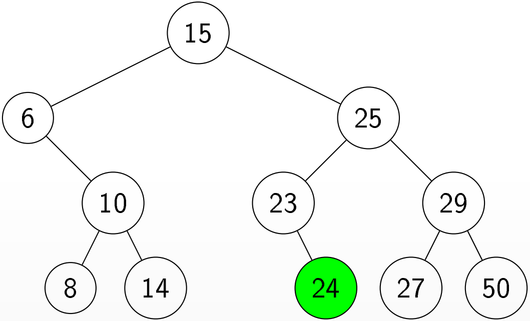


BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

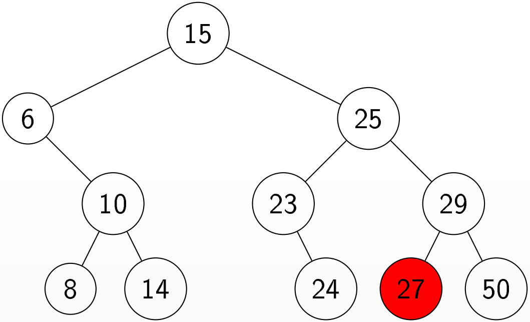
insert(k, v) Search for k, then insert (k, v) as new node

Example: *insert*$(24, \dots)$



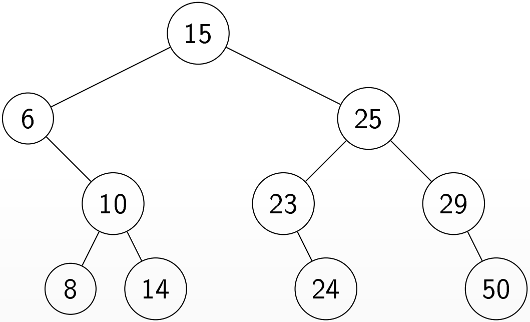
BST Delete

- If node is a leaf, just delete it.



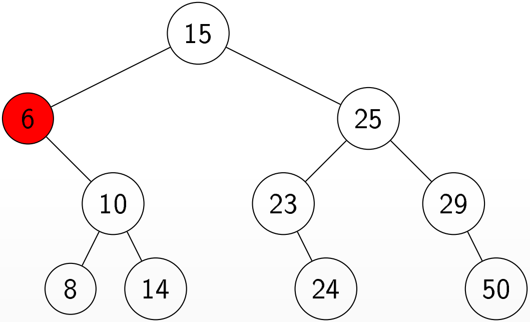
BST Delete

- If node is a leaf, just delete it.



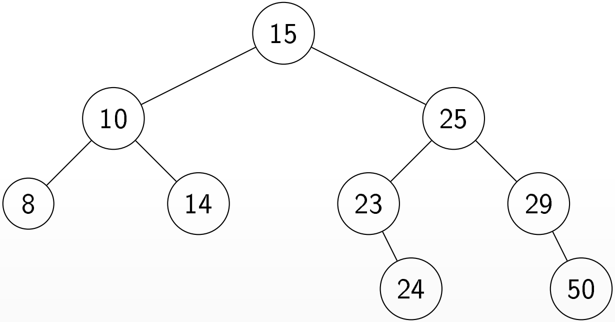
BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



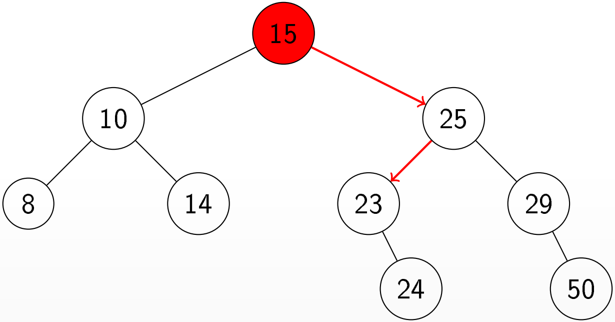
BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



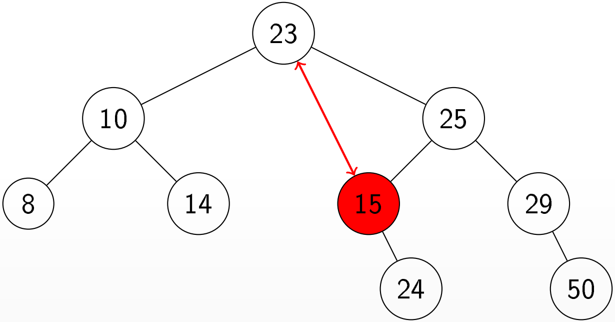
BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up
- Else, swap with *successor* or *predecessor* node and then delete



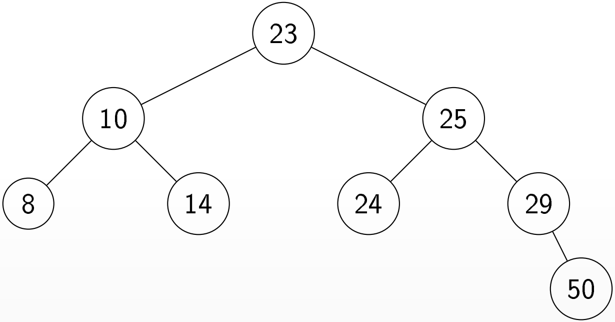
BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up
- Else, swap with *successor* or *predecessor* node and then delete
  - successor and predecssor have one or zero children (why?)



BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up
- Else, swap with *successor* or *predecessor* node and then delete
    - successor and predecssor have one or zero children (why?)



Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case:

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case: $\Theta(n)$
- Best-case:

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case: $\Theta(n)$
- Best-case: $\Theta(\log n)$
- Average-case:

Height of a BST

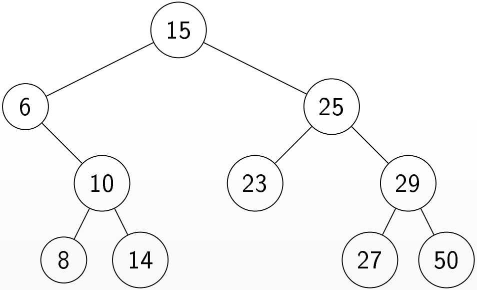
search, *insert*, *delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case: $\Theta(n)$
- Best-case: $\Theta(\log n)$
- Average-case: $\Theta(\log n)$
  (similar analysis to *quick-sort* with random pivot)

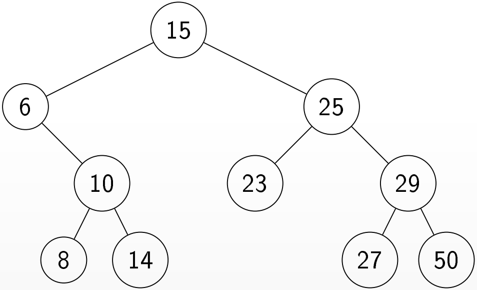
Binary Search Trees

- How to find max/min elements in a BST?



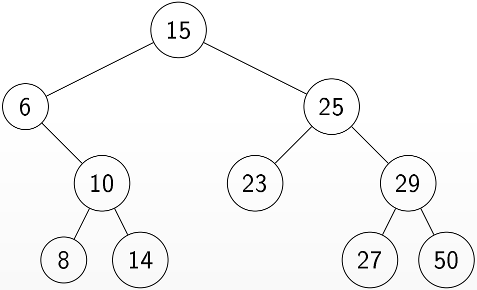
Binary Search Trees

- How to find max/min elements in a BST?
  - Just find the rightmost/leftmost node in $\Theta(h)$ time



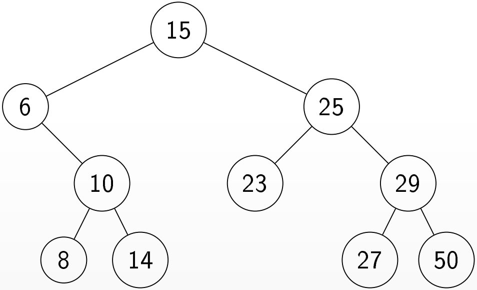
Binary Search Trees

- How to find max/min elements in a BST?
  - Just find the rightmost/leftmost node in $\Theta(h)$ time
- How can I print all keys in sorted order?



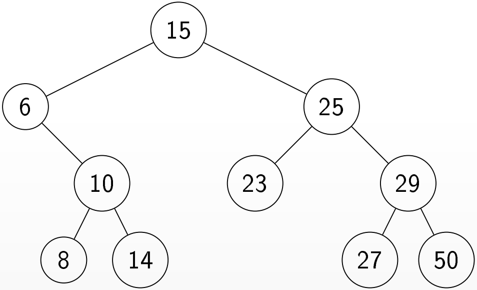
Binary Search Trees

- How to find max/min elements in a BST?
  - Just find the rightmost/leftmost node in $\Theta(h)$ time
- How can I print all keys in sorted order?
  - Do an in-order traversal of the tree in $\Theta(n)$ time
  - Can we do that in $o(n)$?



Binary Search Trees

- How to find max/min elements in a BST?
  - Just find the rightmost/leftmost node in $\Theta(h)$ time
- How can I print all keys in sorted order?
  - Do an in-order traversal of the tree in $\Theta(n)$ time
  - Can we do that in $o(n)$? no! we need to report an output of size n
- **BSTs maintain data in sorted order, which is useful for some queries (an advantage over hash tables which scatter data).**



Balanced BSTs

- Perfectly balanced BSTs: all nodes except for the bottom 2 levels are full (have two children).
    - Too strict for efficient BST balancing.

Balanced BSTs

- Perfectly balanced BSTs: all nodes except for the bottom 2 levels are full (have two children).
    - Too strict for efficient BST balancing.
- Weight balanced: at each internal node i, at least cn_i nodes are in its left subtree and cn_i in its right subtree, for some constant $c \in (0, 1/2]$, where n_i denotes the number of descendants for node i.

Balanced BSTs

- Perfectly balanced BSTs: all nodes except for the bottom 2 levels are full (have two children).
  - Too strict for efficient BST balancing.
- Weight balanced: at each internal node i, at least cn_i nodes are in its left subtree and cn_i in its right subtree, for some constant $c \in (0, 1/2]$, where n_i denotes the number of descendants for node i.
- Height balanced: heights of left and right subtrees of each internal node differ by at most k, for some constant $k \geq 1$.
  - For AVL trees, $k = 1$.
  - We will assume $k = 1$ for the remainder of our discussion.

Balanced BSTs

- Perfectly balanced BSTs: all nodes except for the bottom 2 levels are full (have two children).
    - Too strict for efficient BST balancing.
- Weight balanced: at each internal node i, at least cn_i nodes are in its left subtree and cn_i in its right subtree, for some constant $c \in (0, 1/2]$, where n_i denotes the number of descendants for node i.
- Height balanced: heights of left and right subtrees of each internal node differ by at most k, for some constant $k \geq 1$.
    - For AVL trees, $k = 1$.
    - We will assume $k = 1$ for the remainder of our discussion.
- Height $\Theta(\log n)$ where n is the number of nodes in the tree.

Balanced BSTs

- Perfectly balanced BSTs: all nodes except for the bottom 2 levels are full (have two children).
  - Too strict for efficient BST balancing.
- Weight balanced: at each internal node i, at least cn_i nodes are in its left subtree and cn_i in its right subtree, for some constant $c \in (0, 1/2]$, where n_i denotes the number of descendants for node i.
- Height balanced: heights of left and right subtrees of each internal node differ by at most k, for some constant $k \geq 1$.
  - For AVL trees, $k = 1$.
  - We will assume $k = 1$ for the remainder of our discussion.
- Height $\Theta(\log n)$ where n is the number of nodes in the tree.
- **All balanced BSTs (with respect to any of above definitions) have height $\Theta(\log n)$**
  - We see the proof for height-balanced BSTs in a minute.

Tree height

Definition

The **height** of a node a is the length of the longest path between a and any descendent of a

- as opposed to **depth** which is the length of the path between a and the root.
- Height can be defined recursively as follows:

$$height(a) = \begin{cases} -1, & a = \Phi \\ 1 + max\{height(a.left), height(a.right)\} & a \neq \Phi \end{cases}$$

Tree height

Definition

The **height** of a node a is the length of the longest path between a and any descendent of a

- as opposed to **depth** which is the length of the path between a and the root.
- Height can be defined recursively as follows:

$$height(a) = \begin{cases} -1, & a = \Phi \\ 1 + max\{height(a.left), height(a.right)\} & a \neq \Phi \end{cases}$$

- For a height-balanced BST with $k = 1$, the balancing factor (the difference between the height of the two children) for any node is in $\{-1, 0, 1\}$.

Bounds for the height of height-balanced BSTs

Theorem

For the height $h(n)$ of a height-balanced BST (with $k = 1$) on sufficiently large n nodes we have $\log(n)-1 < h(n) < 1.45\log(n+1)$

- This implies $h(n) \in \Theta(\log n)$.
- Let's see the proof.

Lower Bound for the height of height-balanced BSTs

- We want to prove $\log(n) - 1 < h(n)$.
- The number of nodes in a binary search tree of height h is at most:

$$n \leq 2^{h+1} - 1 \Rightarrow \log n \leq \log(2^{h+1} - 1) < \log(2^{h+1}) = h + 1$$

Hence, we have $\log n - 1 < h$.

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) =$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) =$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) = \ 2$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45\log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) = \ 2 \quad s(2) =$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) = \ 2 \quad s(2) = 4$

$$s(h) = \begin{cases} 1 & h = 0 \\ 2 & h = 1 \\ s(h-1) + s(h-2) + 1, & h \geq 2 \end{cases}$$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) = \ 2 \quad s(2) = 4$

$$s(h) = \begin{cases} 1 & h = 0 \\ 2 & h = 1 \\ s(h-1) + s(h-2) + 1, & h \geq 2 \end{cases}$$

- We can say $s(h) > F(h)$ where $F(h)$ is the h'th Fibonacci number.
  - For large n, we have $F(h) \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{h+1} - 1$

Upper Bound for the height of height-balanced BSTs

- We want to show $h(n) < 1.45 \log(n+1)$.
  - Let $s(h)$ denote the minimum number of nodes in a height-balanced BST (with $k = 1$) of height h.
  - We have $s(0) = 1 \quad s(1) = \ 2 \quad s(2) = 4$

$$s(h) = \begin{cases} 1 & h = 0 \\ 2 & h = 1 \\ s(h-1) + s(h-2) + 1, & h \geq 2 \end{cases}$$

- We can say $s(h) > F(h)$ where $F(h)$ is the h'th Fibonacci number.
  - For large n, we have $F(h) \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+1} - 1$

We have $n > \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{h+1} - 1 \rightarrow \sqrt{5}(n+1) \geq (\frac{1+\sqrt{5}}{2})^{h+1} \rightarrow$
$\log(\sqrt{5}(n+1)) \geq (h+1)\log(\frac{1+\sqrt{5}}{2}) \rightarrow h < \frac{\log\sqrt{5} + \log(n+1)}{\log(1+\sqrt{5}) - 1} - 1$
$= \frac{1}{\log(1+\sqrt{5})-1}\log(n+1) + \frac{\log\sqrt{5}}{\log(1+\sqrt{5})-1} - 1 < 1.45\log(n+1)$

Bounds for the height of height-balanced BSTs

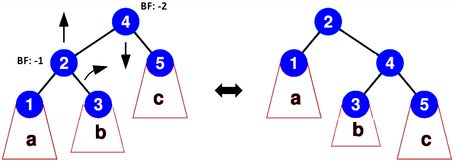
Theorem

For the height $h(n)$ of a height-balanced BST (with $k = 1$) on sufficiently large n nodes we have $\log(n)-1 < h(n) < 1.45\log(n+1)$

- This implies $h(n) \in \Theta(\log n)$.
- So, it is desirable to maintain a height-balanced binary search tree (they are asymptotically the best possible BSTs).

BST Single Rotation

- Height of a height-balanced BST on n nodes is $\Theta(\log n)$
- A **self-balancing BST** maintains the height-balanced property after an insertion/deletion via **tree rotation**



- Every rotation swaps parent-child relationship between two nodes (here between 2 and 4)
- Tree rotation preserves the BST key ordering property.
- Each rotation requires updating a few pointers in $O(1)$ time.
- original height: $\max(\mathit{height}(a)+2, \mathit{height}(b)+2, \mathit{height}(c)+1)$
  new height: $\mathit{max}(\mathit{height}(a)+1; \mathit{height}(b)+2; \mathit{height}(c)+2)$

AVL Trees

- Introduced by Adelson-Velskiĭ and Landis in 1962
- An *AVL Tree* is a height-balanced BST
  - The heights of the left and right subtree differ by at most 1.
  - (The height of an empty tree is defined to be -1.)
- At each non-empty node, we store $\mathit{height}(R) - \mathit{height}(L) \in \{-1, 0, 1\}$:

  -1 means the tree is *left-heavy*
  0 means the tree is *balanced*
  1 means the tree is *right-heavy*
- We could store the actual height, but storing balances is simpler and more convenient.

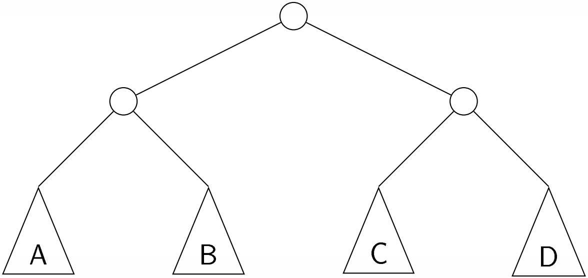
AVL insertion

To perform *insert*(T, k, v):

- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1, 0, or 1, then keep going.
- If the balance factor is ± 2, then call the *fix* algorithm to "rebalance" at that node.

How to “fix” an unbalanced AVL tree

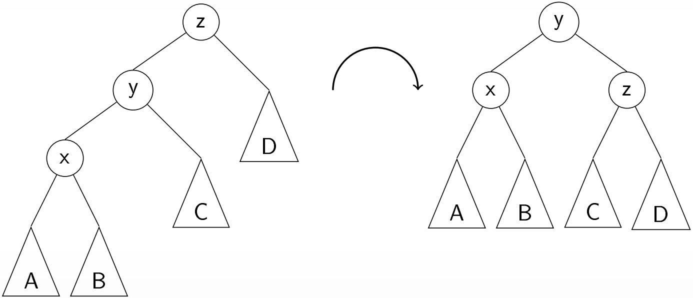
Goal: change the *structure* without changing the *order*



Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.

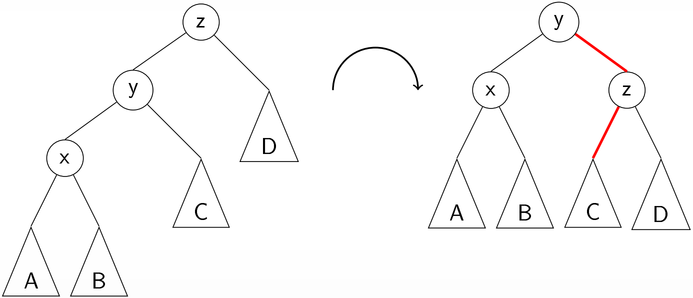
Right Rotation

- When the followings hold, we apply a **right rotation** on node z
  - The balance factor at z is -2.
  - The balance factor of y is 0 or -1.



Right Rotation

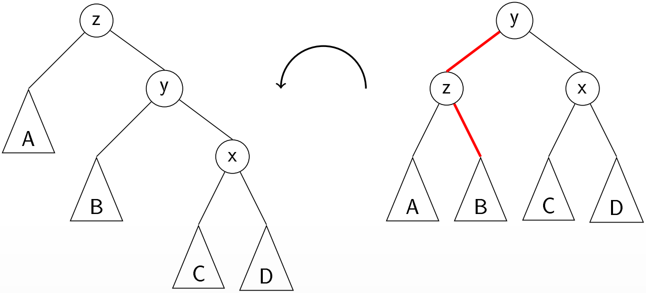
- When the followings hold, we apply a **right rotation** on node z
  - The balance factor at z is -2.
  - The balance factor of y is 0 or -1.



Note: Only two edges need to be moved, and two balances updated.

Left Rotation

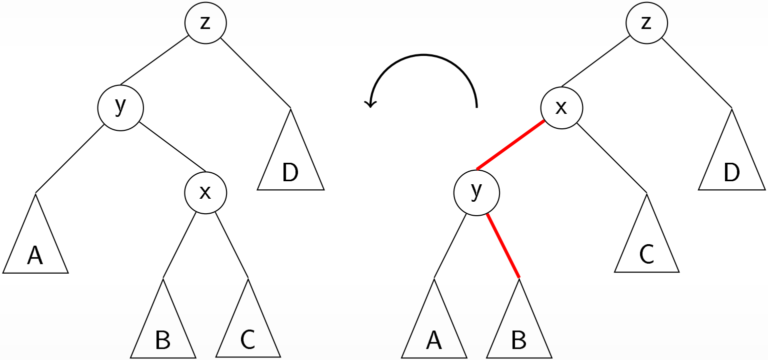
- When the followings hold, we apply a **left rotation** on node z
  - The balance factor at z is 2.
  - The balance factor of y is 0 or 1.



Again, only two edges need to be moved and two balances updated.

Double Right Rotation

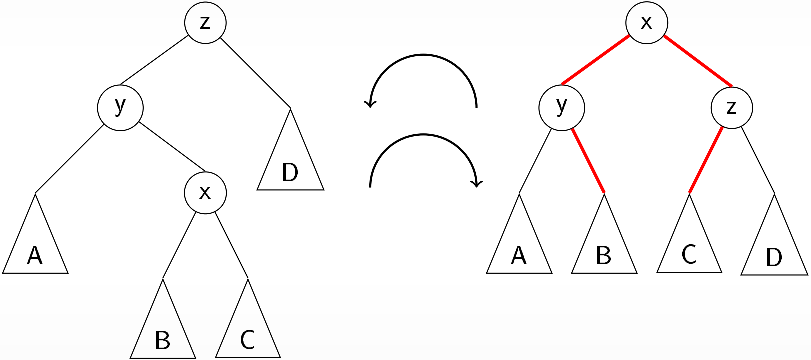
- When the followings hold, we apply a **double right rotation** on z
  - The balance factor at z is -2 & the balance factor of y is 1.



- First, a left rotation on the left subtree (y).

Double Right Rotation

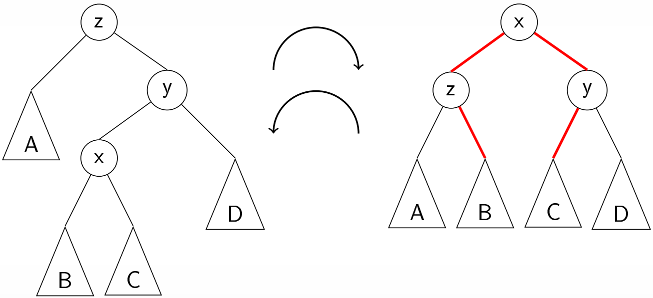
- When the followings hold, we apply a **double right rotation** on z
  - The balance factor at z is -2 & the balance factor of y is 1.



- First, a left rotation on the left subtree (y).
- Second, a right rotation on the whole tree (z).

Double Left Rotation

This is a *double left rotation* on node z; apply when balance of z is 2 and balance of y is -1.



Right rotation on right subtree (y),
followed by left rotation on the whole tree (z).

AVL Tree Operations

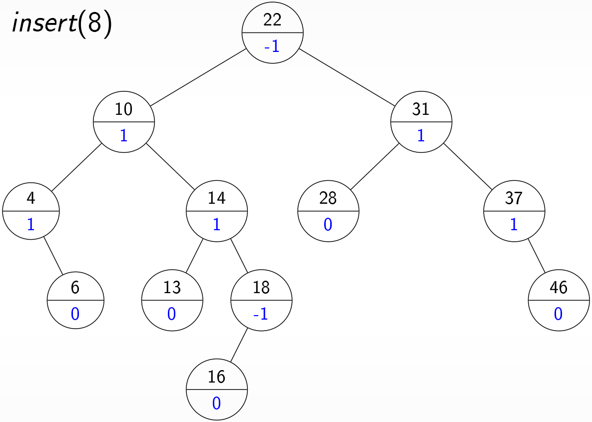
search: Just like in BSTs, costs $\Theta(\mathit{height})$

insert: Shown already, total cost $\Theta(\mathit{height})$
fix will be called *at most once*.

delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).
fix may be called $\Theta(\mathit{height})$ times.
Total cost is $\Theta(\mathit{height})$.

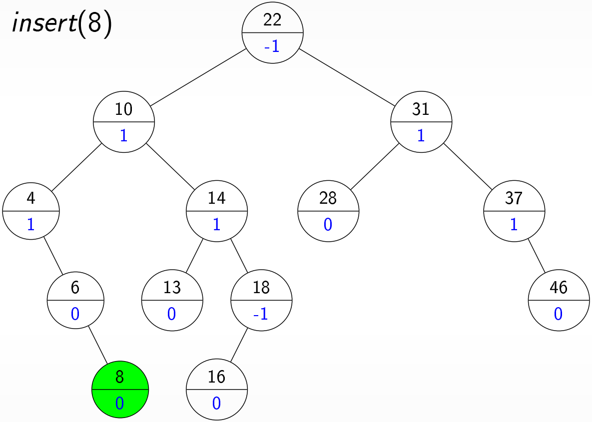
AVL tree examples

Example: *insert*(8)



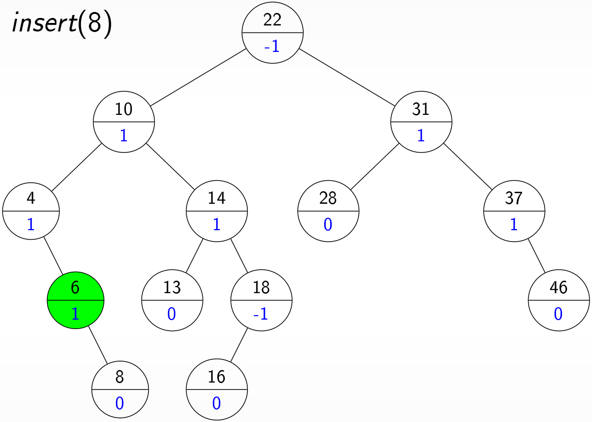
AVL tree examples

Example: *insert*(8)



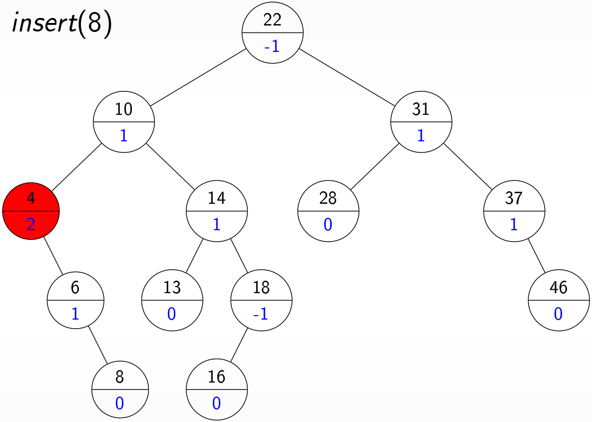
AVL tree examples

Example: *insert*(8)



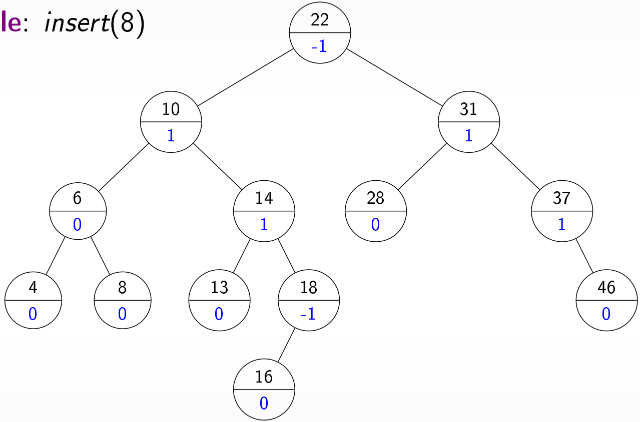
AVL tree examples

Example: *insert*(8)



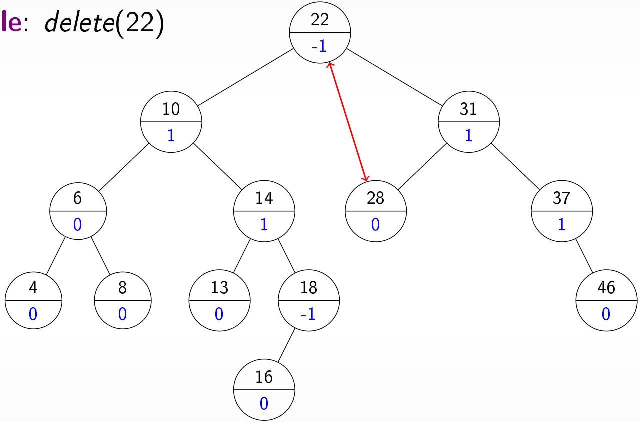
AVL tree examples

Example: *insert*(8)



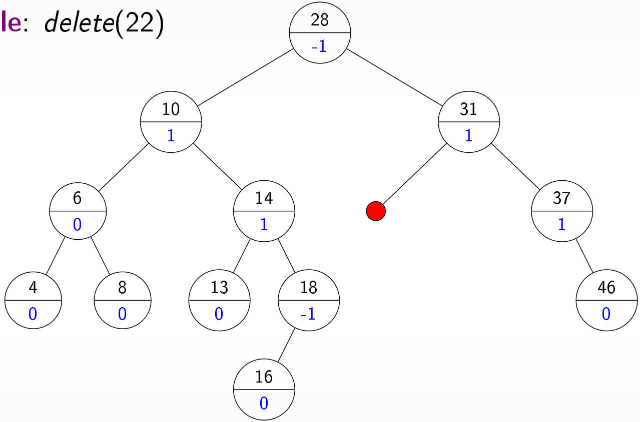
AVL tree examples

Example: *delete*(22)



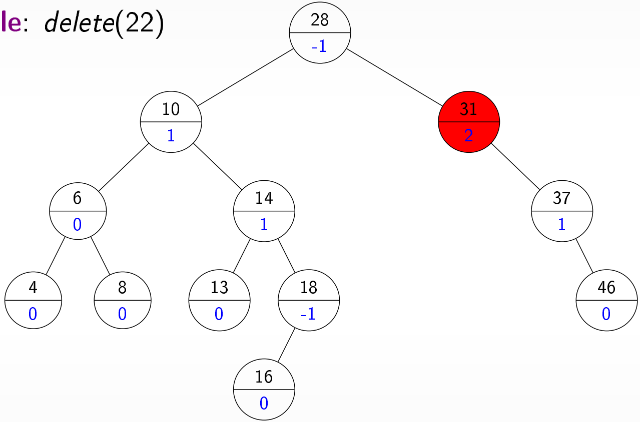
AVL tree examples

Example: *delete*(22)



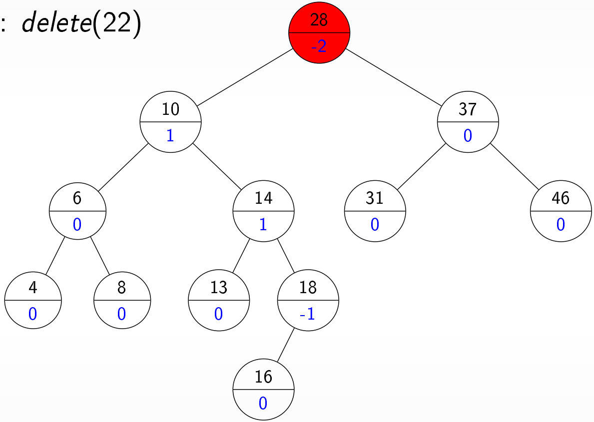
AVL tree examples

Example: *delete*(22)



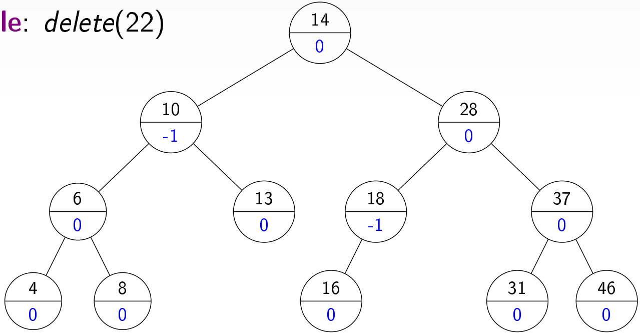
AVL tree examples

Example: *delete*(22)



AVL tree examples

Example: *delete*(22)



AVL tree analysis

- Since AVL-trees are height-balanced, their height is $\Theta(\log n)$
- Search can be done as before (no need for rebalancing)
- Insert(x) takes $\Theta(\log n)$ and involves at most one fix.

AVL tree analysis

- Since AVL-trees are height-balanced, their height is $\Theta(\log n)$
- Search can be done as before (no need for rebalancing)
- Insert(x) takes $\Theta(\log n)$ and involves at most one fix.
- Delete(x) takes $\Theta(\log n)$ and involves at most $\Theta(\log n)$ fixes.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

AVL tree analysis

- Since AVL-trees are height-balanced, their height is $\Theta(\log n)$
- Search can be done as before (no need for rebalancing)
- Insert(x) takes $\Theta(\log n)$ and involves at most one fix.
- Delete(x) takes $\Theta(\log n)$ and involves at most $\Theta(\log n)$ fixes.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

- What about other queries (e.g., get-max(), get-min(), rank(), select())?

AVL tree analysis

- Since AVL-trees are height-balanced, their height is $\Theta(\log n)$
- Search can be done as before (no need for rebalancing)
- Insert(x) takes $\Theta(\log n)$ and involves at most one fix.
- Delete(x) takes $\Theta(\log n)$ and involves at most $\Theta(\log n)$ fixes.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

- What about other queries (e.g., get-max(), get-min(), rank(), select())?
- One great thing about AVL trees is that they can be easily augmented to support these queries in a good time (this is the main advantage of the trees over say Hash tables).

Augmented Data Structures

- In practice, it often happens that you want an abstract data type to support additional queries
  - To implement this, we need to **augment** the underlying data structure
  - Augmentation often involves storing additional data which facilitates the query.

Augmented Data Structures

- In practice, it often happens that you want an abstract data type to support additional queries
  - To implement this, we need to **augment** the underlying data structure
  - Augmentation often involves storing additional data which facilitates the query.
- Consider AVL tree which supports search, insert, delete in $\Theta(\log n)$ time
  - What if your 'boss' asks you to **additionally** support minimum, maximum, rank, and select?

Augmented Data Structures

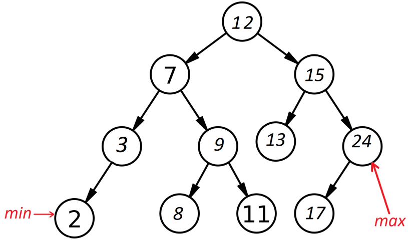
- In practice, it often happens that you want an abstract data type to support additional queries
  - To implement this, we need to **augment** the underlying data structure
  - Augmentation often involves storing additional data which facilitates the query.
- Consider AVL tree which supports search, insert, delete in $\Theta(\log n)$ time
  - What if your 'boss' asks you to **additionally** support minimum, maximum, rank, and select?
  - Without augmentation, minimum and maximum take $\Theta(\log n)$ while rank and select require linear time (in-order traversal to retrieve the sorted list of keys).
  - What if your boss wants them to be faster?

Augmenting Data Structures

- First, figure out what additional information should be store?
- Second, figure out how, using the additional information, answer new queries (e.g., min and rank in AVL trees) efficiently?
- Third, figure out how to update existing operations (e.g., insertion and deletion) to keep the stored information updated.

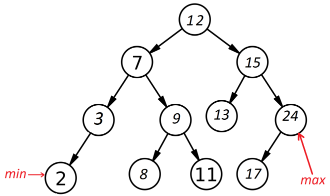
Augmenting AVL trees

- We can augment AVL trees to support **minimum/maximum** in $\Theta(1)$.
- Just add a pointer to the leftmost/rightmost leaf of the tree.
- After updating the tree by an insert/deleted, make sure that the pointer still points to the smallest/largest element



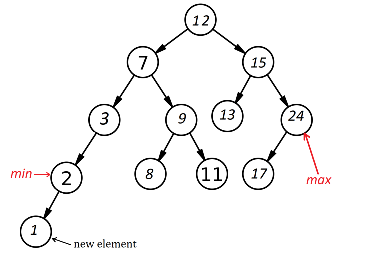
Augmenting AVL trees

- After an insertion, first, re-arrange the tree if required (to keep it AVL). Keep a pointer to the newly inserted element
    - After the insertion, if the newly inserted key is less than minimum, update the the minimum pointer to point to it (similar for maximum pointer).



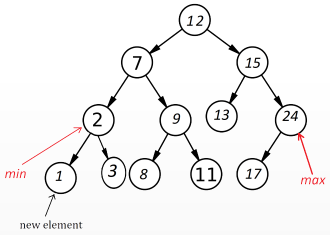
Augmenting AVL trees

- After an insertion, first, re-arrange the tree if required (to keep it AVL). Keep a pointer to the newly inserted element
  - After the insertion, if the newly inserted key is less than minimum, update the the minimum pointer to point to it (similar for maximum pointer).



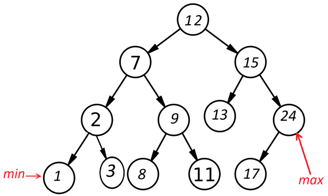
Augmenting AVL trees

- After an insertion, first, re-arrange the tree if required (to keep it AVL). Keep a pointer to the newly inserted element
  - After the insertion, if the newly inserted key is less than minimum, update the the minimum pointer to point to it (similar for maximum pointer).



Augmenting AVL trees

- After an insertion, first, re-arrange the tree if required (to keep it AVL). Keep a pointer to the newly inserted element
  - After the insertion, if the newly inserted key is less than minimum, update the the minimum pointer to point to it (similar for maximum pointer).



Augmenting AVL trees

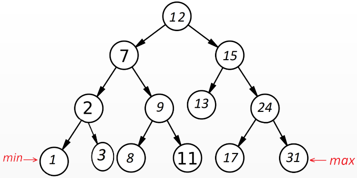
- After an insertion, first, re-arrange the tree if required (to keep it AVL). Keep a pointer to the newly inserted element
  - After the insertion, if the newly inserted key is less than minimum, update the the minimum pointer to point to it (similar for maximum pointer).
  - It takes an additional time of $\Theta(1)$ (the insertion time is still $\Theta(\log n)$).
- Similar update for max pointer

Augmenting AVL trees

- For deleting node x, check if x is the minimum element. If so, first update the minimum pointer to the successor of x.

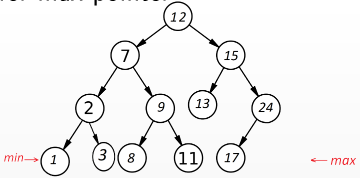
Augmenting AVL trees

- For deleting node x, check if x is the minimum element. If so, first update the minimum pointer to the successor of x.
- Finding the successor of minimum takes additional time of $\Theta(1)$
  - Let x be the min element before deletion; we know there is nothing on the left of x.
  - The right subtree of x has zero or one node (otherwise x is unbalanced).
  - If there is an item y on the right of x, then it is the successor of x
  - If y is a leaf, then its parent is the successor



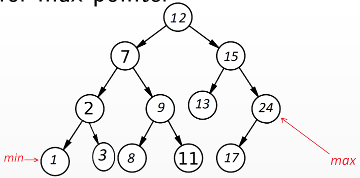
Augmenting AVL trees

- For deleting node x, check if x is the minimum element. If so, first update the minimum pointer to the successor of x.
- Finding the successor of minimum takes additional time of $\Theta(1)$
  - Let x be the min element before deletion; we know there is nothing on the left of x.
  - The right subtree of x has zero or one node (otherwise x is unbalanced).
  - If there is an item y on the right of x, then it is the successor of x
  - If y is a leaf, then its parent is the successor
- After updating the pointer, delete as in regular AVL trees.
- Similar update for max pointer



Augmenting AVL trees

- For deleting node x, check if x is the minimum element. If so, first update the minimum pointer to the successor of x.
- Finding the successor of minimum takes additional time of $\Theta(1)$
    - Let x be the min element before deletion; we know there is nothing on the left of x.
    - The right subtree of x has zero or one node (otherwise x is unbalanced).
    - If there is an item y on the right of x, then it is the successor of x
    - If y is a leaf, then its parent is the successor
- After updating the pointer, delete as in regular AVL trees.
- Similar update for max pointer



Augmenting AVL trees

Theorem

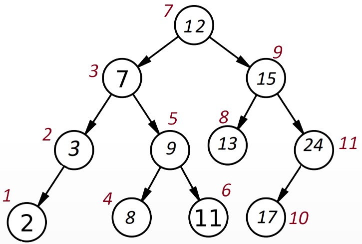
We can augment AVL trees by adding only two pointers ($\Theta(1)$) extra space to support minimum/maximum queries in $\Theta(1)$ and without changing time complexity of other queries (insertion, deletion, and search).

Augmenting AVL trees

- Can we augment AVL trees to support rank/select operations in $O(\log n)$ time?
  - *rank*(x) reports the index of key x in the sorted array of keys
  - *select*(i) returns the key with index i in the sorted array of keys

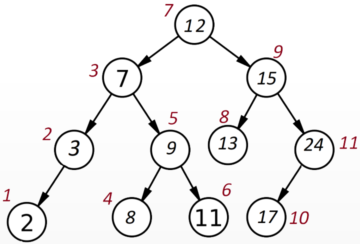
Augmenting AVL trees

- Can we augment AVL trees to support rank/select operations in $O(\log n)$ time?
  - $rank(x)$ reports the index of key x in the sorted array of keys
  - $select(i)$ returns the key with index i in the sorted array of keys
- Idea 1: Store the rank of each node at that node.
  - $O(\log n)$ rank and select are guaranteed (why?)
  - Is it a good augment data structure?



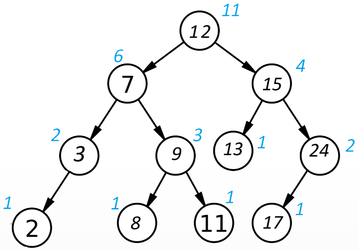
Augmenting AVL trees

- Can we augment AVL trees to support rank/select operations in $O(\log n)$ time?
  - *rank*(x) reports the index of key x in the sorted array of keys
  - *select*(i) returns the key with index i in the sorted array of keys
- Idea 1: Store the rank of each node at that node.
  - $O(\log n)$ rank and select are guaranteed (why?)
  - Is it a good augment data structure? No because inserting an item (e.g., key 1 here) might require updating all stored ranks Insertion/deletion take $\Theta(n)$. Failed!



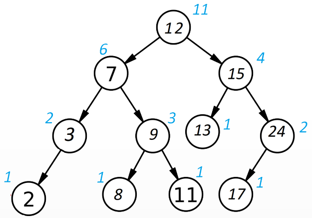
Augmenting AVL trees

- Idea 2: At each node, store the size (no. of nodes))of the subtree rooted at that node
  - The size of a node is the sum of the sizes of its two subtrees plus 1.
  - The size of an empty subtree is 0.
- The rank of a node x **in its own subtree** is the size of its left subtree.



Selection in Augmented AVL trees

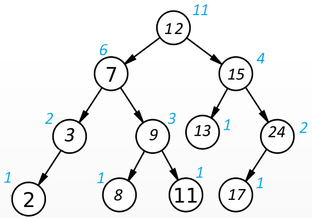
- Selection on an AVL tree augmented with size data is similar to quickselect, where the root acts as a pivot.
- Select(i): compare i with the rank of the root r (size of left subarray).
  - If equal, return the root r
  - if $i <$ rank(root), recursively find the same index i in the left subtree
  - if $i >$ rank(root), recursively find index $i - rank(root) - 1$ in the right subtree



Selection in Augmented AVL trees

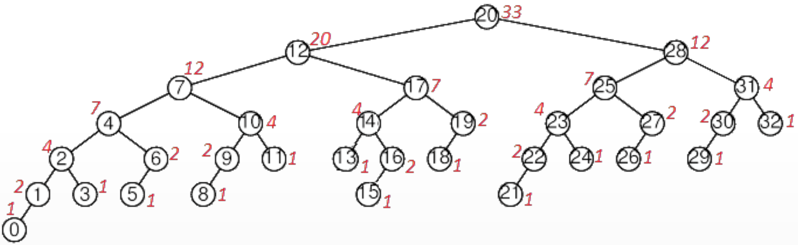
- Selection on an AVL tree augmented with size data is similar to quickselect, where the root acts as a pivot.
- Select(i): compare i with the rank of the root r (size of left subarray).
  - If equal, return the root r
  - if $i <$ rank(root), recursively find the same index i in the left subtree
  - if $i >$ rank(root), recursively find index $i - rank(root) - 1$ in the right subtree

E.g., select(5,12) $\xrightarrow{left}$ select(5,7) $\xrightarrow{right}$ select(2,9) $\xrightarrow{right}$ select(0,11) $\xrightarrow{equal}$ 11 is returned



Augmenting AVL trees

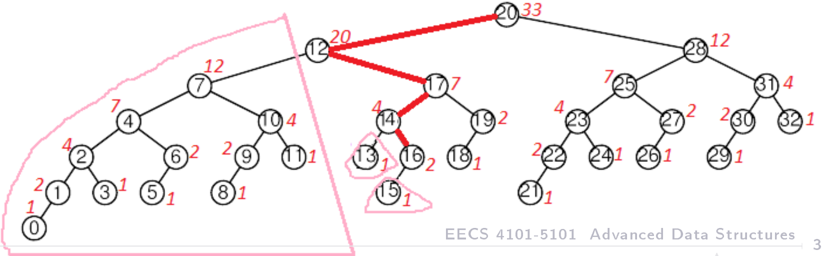
- To find **rank(x)** on an AVL tree augmented, search for k.
- On the path from the root to x, sum up sizes of all left sub trees
    - When searching for x, when you recurs on the right subtree, add up the size of the left subtree plus one (for the current node).
    - When the node was found, add up the size of its left subtree to the computed rank.



Augmenting AVL trees

- To find **rank(x)** on an AVL tree augmented, search for k.
- On the path from the root to x, sum up sizes of all left sub trees
  - When searching for x, when you recurs on the right subtree, add up the size of the left subtree plus one (for the current node).
  - When the node was found, add up the size of its left subtree to the computed rank.

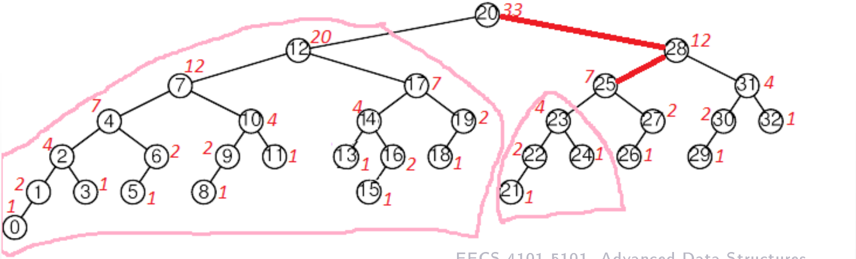
rank(16,20) $\xrightarrow{\textit{left}}$ rank(16,12) res += 12+1 $\xrightarrow{\textit{right}}$ rank(16,17) $\xrightarrow{\textit{left}}$ rank(16,14) res+= 1+1 $\xrightarrow{\textit{right}}$ rank(16,16) res+= 1



Augmenting AVL trees

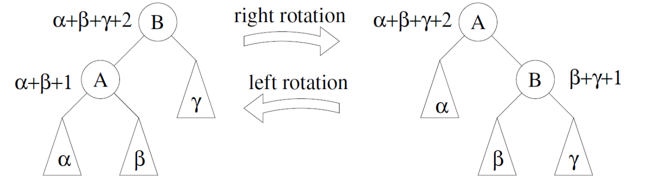
- To find **rank(x)** on an AVL tree augmented, search for k.
- On the path from the root to x, sum up sizes of all left sub trees
  - When searching for x, when you recurs on the right subtree, add up the size of the left subtree plus one (for the current node).
  - When the node was found, add up the size of its left subtree to the computed rank.

rank(25,20) res+= 20+1 $\xrightarrow{right}$ rank(25,28) $\xrightarrow{left}$ rank(25,25) res += 4.



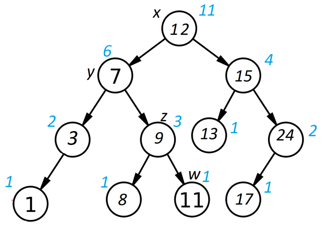
Updating Augmented AVL trees

- After an **insertion**, the sizes of all ancestors of the new node should be incremented; do it before fixing the tree.
- After a **deletion**, the sizes of all ancestors of the deleted node should be decremented; do it before fixing the tree.
- The 2 nodes involved in each **single rotation** must have their sizes updated. (recall that double rotation involves two single rotations)
    - Only sizes of A and B should be updated. It can be done in constant time!



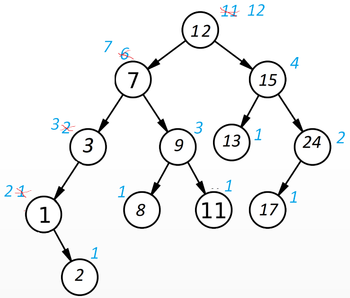
Updating Augmenting AVL trees

- insert(2): first insert the new node and update sizes of ancestors.



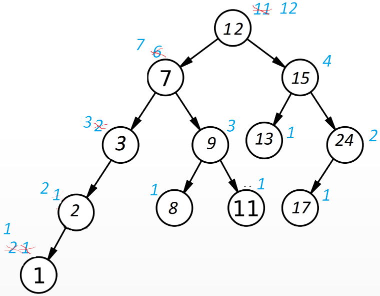
Updating Augmenting AVL trees

- insert(2): first insert the new node and update sizes of ancestors.
- After the insertion, node 3 is unbalanced, since it is left-heavy and its left child (1) is right heavy, first apply a left rotation; update the sizes of the two involved node (1 and 2).



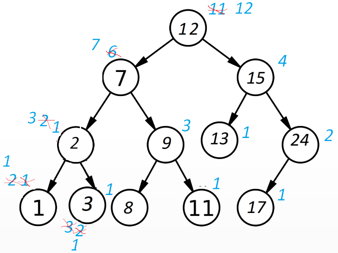
Updating Augmenting AVL trees

- insert(2): first insert the new node and update sizes of ancestors.
- After the insertion, node 3 is unbalanced, since it is left-heavy and its left child (1) is right heavy, first apply a left rotation; update the sizes of the two involved node (1 and 2).
- Now 3 is left-heavy and its left child (2) is not right-heavy; apply a single rotation between them and update their sizes



Updating Augmenting AVL trees

- insert(2): first insert the new node and update sizes of ancestors.
- After the insertion, node 3 is unbalanced, since it is left-heavy and its left child (1) is right heavy, first apply a left rotation; update the sizes of the two involved node (1 and 2).
- Now 3 is left-heavy and its left child (2) is not right-heavy; apply a single rotation between them and update their sizes



Augmenting AVL trees

Theorem

It is possible to augment an AVL tree by storing the sizes of each subtree so that select and rank operations can be supported in $\Theta(\log n)$ time. The time complexity of other operations (search, insert, and delete) remain unchanged.

- In fact, we can merge such AVL tree with a doubly linked list to support predecessor and successor operations.

Augmented Data Structures Summary

- Steps to Augmenting a Data Structure
  - Specify an ADT (including additional operations to support).
  - Choose an underlying data structure.
  - Determine the additional data to be maintained.
  - Develop algorithms for new operations.
  - Verify that the additional data can be maintained efficiently during updates.