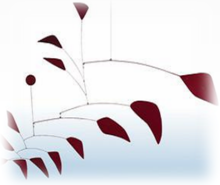
EEEC 4101-5101
Advanced Data Structures



Shahin Kamali

Topic 1b - Amortized Analysis

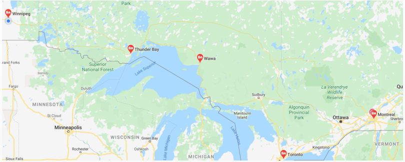
CLRS 17-1, 17-2, 17-3, 17-4

York University

Picture is from the cover of the textbook CLRS.

Amortized vs Average Analysis

- Assume you want to drive from Winnipeg to Montreal (distance is $\approx 2600km$).



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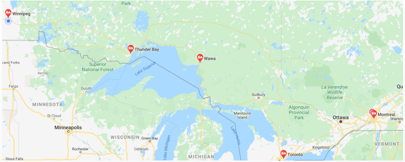

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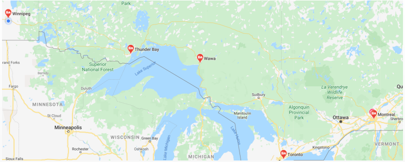

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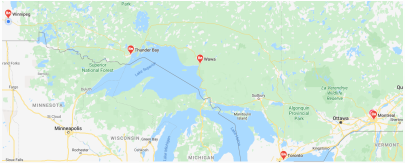

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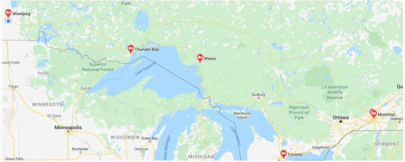

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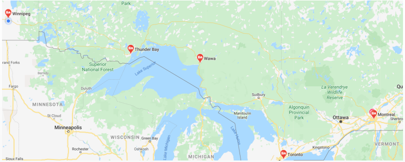
- On average, you drive $2600/4 = 650$km per day. We say **amortized** distance moved every day is 650km.



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- Average case analysis relies on probabilistic assumptions about the input or the data structure
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  - The worst-case might be met with some small chance (you can be 'lucky' or not).
- Amortized analysis consider consider a **sequence** of consecutive operations.
  - Bound the **total cost** for m operations
  - This gives the amortized cost $B(n)$ **per operation**
  - The amortized cost is only a function of n, the size of stored data
  - Unlike average case analysis, there is no probability distribution
  - Every sequence of m operations is guaranteed to have worst-case time at most $mB(n)$, regardless of the input or the sequence of operations (regardless of how lucky you are).

Amortized vs Average Analysis

- Let's compare two algorithms A and B
- A performs operations which take $\Theta(n)$ time in the worst case and $\Theta(\log n)$ **on average**.
- B performs operations which take $\Theta(n)$ time in the worst case and **amortized** $\Theta(\log n)$.

	worst-case time per operation	average/amortized time per operation	worst-case time for m operations	average time for m operations
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- Start from an initial configuration where all bits are ‘0’
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- Each operation increments the encoded number
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- The i'th bit from right is flipped iff all $i-1$ bits on its right are 1 before the increment ($i \geq 0$)
    - After the flip all bits on the right will be 0.
    - In the next $2^i - 1$ operations after the flip the bit is not flipped.
    - The i'th bit is flipped once in 2^i operations

		$\lceil \log m \rceil$	...		2	1	0		
0	0	0	0	0	0	0	0	initial configuration	
0	0	0	0	0	0	0	1	after 1st increment	1 bit flipped
0	0	0	0	0	0	1	0	after 2nd increment	2 bits flipped
			⋮					⋮	⋮
0	1	1	0	1	1	1	1	after 111th increment	1 bit flipped
0	1	1	1	0	0	0	0	after 112th increment	5 bits flipped

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- For a sequence of m operations, the i'th bit is flipped $\frac{m}{2^i}$ times
- Total number of flips will be at most

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- **The amortized number of flips per operation is $2 = \Theta(1)$ flips**.
- The worst case number of flips is $\Theta(\log m)$; but it never happens that a sequence of m operations have $m\Theta(\log m)$ flips!

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  - Some operations are more 'expensive' and most are 'inexpensive'.
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  - Amortized cost is the average cost over all operations
  - There is no probability distribution or randomness
- We saw the amortized number of flips when incrementing a number m times is $\Theta(1)$
  - Some increment operation need $\Theta(\log m)$ flips while most operation take less flips.
  - On average, each operation needs $\Theta(1)$ flips.

Methods for Amortized Analysis

- There are three frameworks for amortized analysis.
- **Aggregate method**:
  - Sum the total cost of m operations
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- Let's review these methods with an example!

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- Problem: implement a stack stored in an array to support push (insert) operations.
- The problem is **online** in the sense that we do not know how many operations to expect
- How large the array should be? there is a trade-off:
  - larger array: less likely to run out of space, more unused/wasted memory
  - smaller array: more likely to run out of space, less unused/wasted memory

Dynamic Arrays

- Possible solution: maintain arrays with sizes that are powers of 2.
- If the array runs out of space ($n > a$):
  - allocate a new array of size a $2n$
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i	operation

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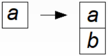

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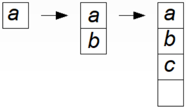

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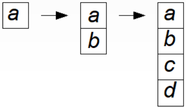

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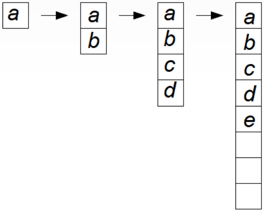

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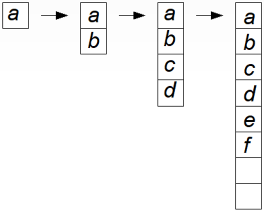

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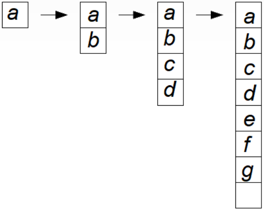

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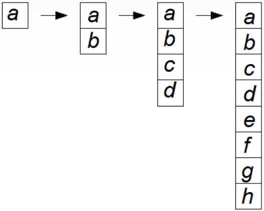

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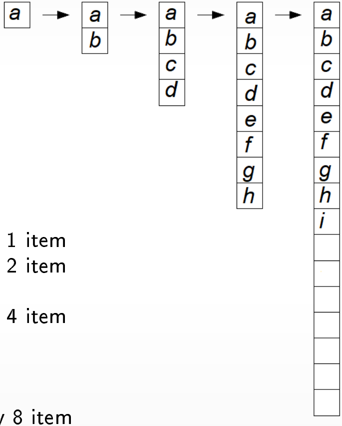

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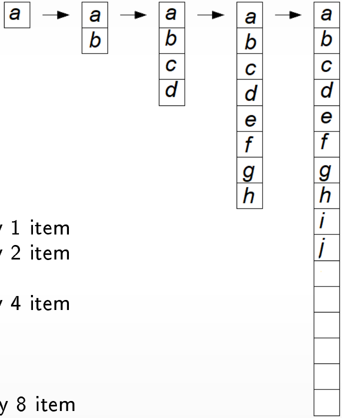

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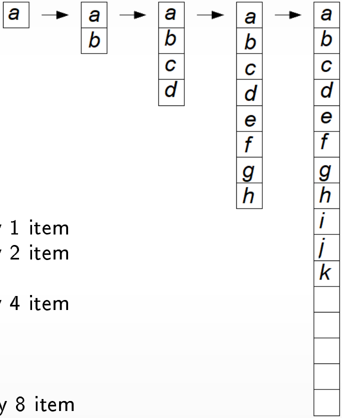

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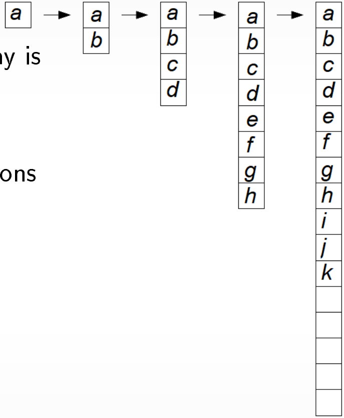

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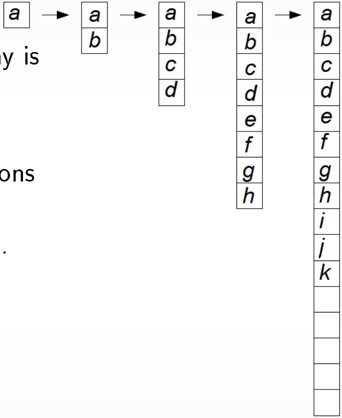

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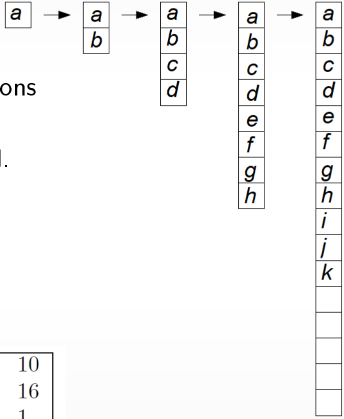
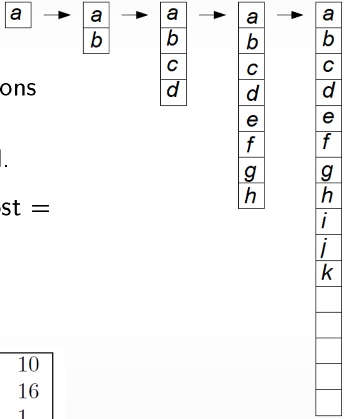


i	1	2	3	4	5	6	7	8	9	10
array size (a)	1	2	4	4	8	8	8	8	16	16
$c(i)$	**1**	**2**	**3**	1	**5**	1	1	1	**9**	1

Dynamic Arrays

- The worst-case cost occurs when the whole array is copied to a new array:
  - $\Theta(n)$ worst-case time per insert.
- Rough estimate: a sequence of m insert operations takes $O(m \cdot n)$ time.
  - We can obtain a much better (smaller) bound.
- Let $c(i)$ denote the cost of the ith insertion (cost = number of insert/copies).

$$c(i) = \begin{cases} i & \text{if } i = 2^k + 1 \text{ for some integer } k \\ 1 & \text{if otherwise} \end{cases}$$



i	1	2	3	4	5	6	7	8	9	10
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Aggregation for Dynamic Arrays

- Aggregate method: find total cost of m operations and divide by m

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&= m + 2^{\lfloor \log(m-1) \rfloor + 1} - 1 \\
&\leq m + 2^{\log m + 1} - 1 \\
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- The amortized cost is hence $\frac{\Theta(m)}{m} = \Theta(1)$
- The aggregate is useful for simple amortized analysis.
- Sometimes require a different technique to obtain amortized cost.

Accounting Method

- Assume you want to prove that your average (amortized) daily cost is no more than 100$.

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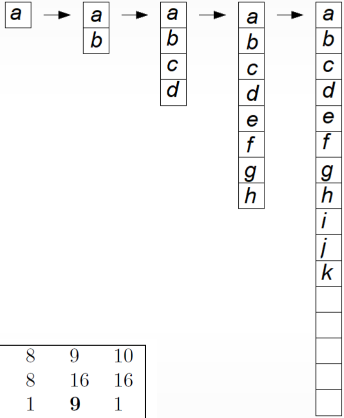
- On days which you spend more than 100$, you should use accumulated credit from previous days
- If your balance remains non-negative at the end of each day, your amortized cost is at most 100$
  - In m consecutive days your expenditure has been at most $100m \rightarrow$ amortized cost at most 100$.

Accounting Method

- Accounting method overview:
  - Each operation **deposits** a fixed credit into an account (This amount is an upper bound on the amortized cost.)
  - Each operation uses 'credit' to pay its **cost**
  - Inexpensive operations save more than their cost
  - Expensive operations cost more more than they save
  - Account must remain non-negative

Accounting Method - Dynamic Arrays

- We prove the amortized cost for insertion is 3



i	1	2	3	4	5	6	7	8	9	10
array size (a)	1	2	4	4	8	8	8	8	16	16
$c(i)$	**1**	**2**	**3**	1	**5**	1	1	1	**9**	1
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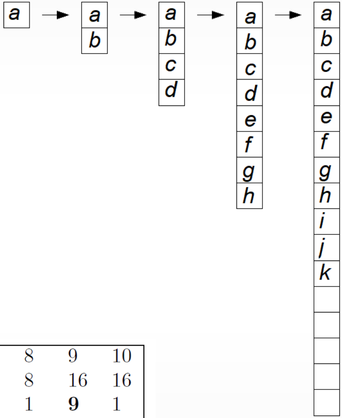

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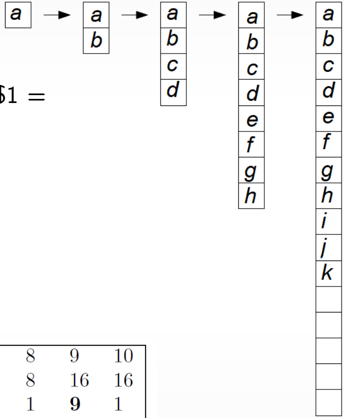

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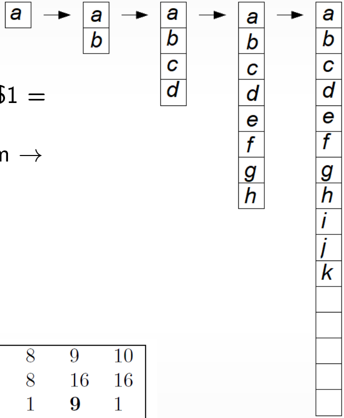

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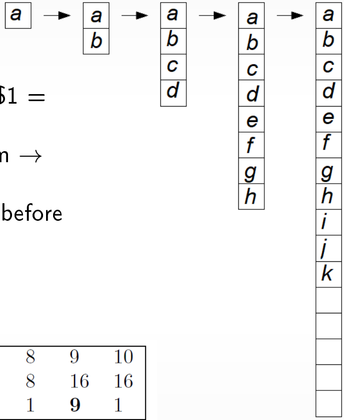

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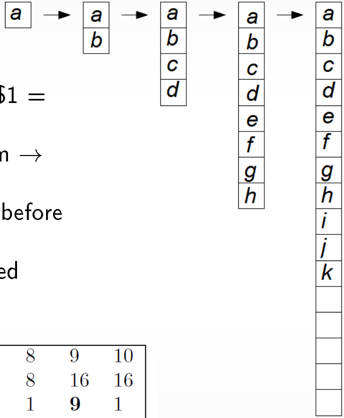

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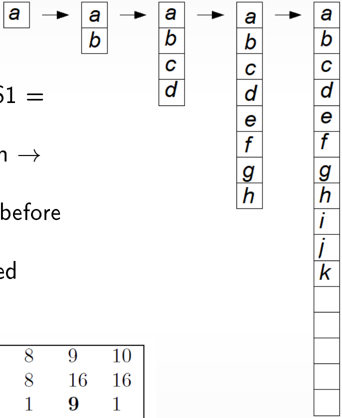
  - → account remains non-negative



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Potential method

- Define a potential function Φ that maps the state of the structure and the index of an operation to an integer
  - Potential is basically the available credit in accounting method

$$\hat{c}(i) = c(i) + \Phi(i) - \Phi(i-1)$$

  - $\hat{c}(i) \rightarrow$ amortized cost of operation i
  - $c(i) \rightarrow$ actual cost of operation i
- Total amortized cost will be total cost plus a constant independent of m.

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    $= i + 2i - 2(i-1) - 2i + 2 + (i-1) = 3$
- **Potential method is often the strongest method for amortized analysis**

Methods for Amortized Analysis

- There are three frameworks for amortized analysis.
- **Aggregate method**:
  - Sum the total cost of m operations
  - Divide by m to get the amortized cost
- **Accounting method**
  - Analogy with a **bank account**, where there are **fixed deposits** and variable withdrawals
- **Potential method**
  - Define amortized cost through **potential function** which maps the sequence of operations to an integer
- Let's review these methods with another example!

Special Stacks

- Consider a stack with one operation $Op(n, x)$, where $n \geq 0$.

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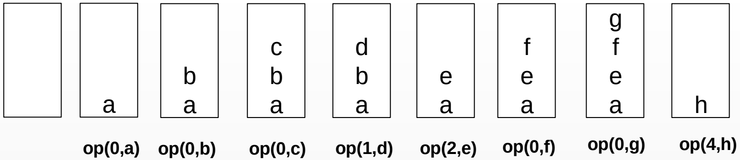

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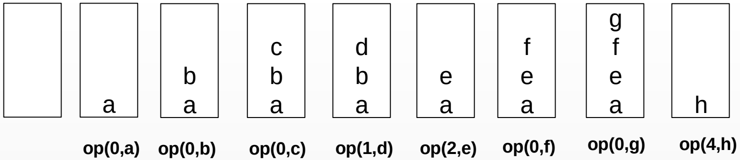
  - Assume each single push and pop has cost 1 (e.g., stack is implemented using a linked list).



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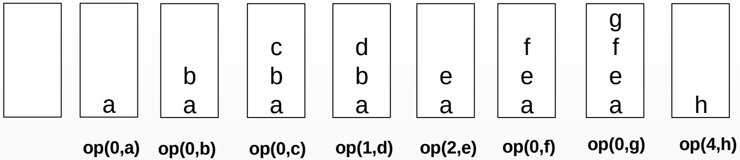

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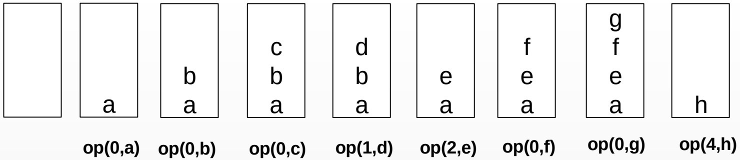
  - A single operation can have a cost of $\Theta(m)$ in the worst case.
  - The amortized time is much better!



Aggregate Method for Special Stacks

- Review of aggregate method:
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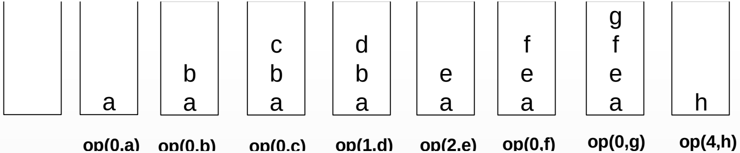
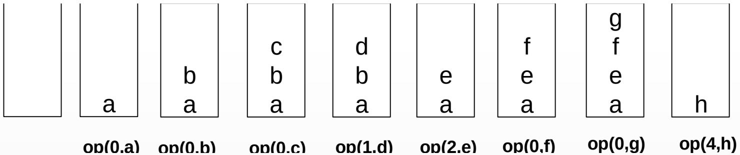
  - Divide by m to get the amortized cost



Aggregate Method for Special Stacks

- Review of aggregate method:
  - Sum the total cost of m consecutive operations
  - Divide by m to get the amortized cost
- Unlike bit flips and dynamic arrays, we cannot predict the cost of the i'th operation.
- The aggregate method is limited and cannot help for amortized analysis of special stacks!



Accounting Method for Special Stacks

- Review of accounting method:
  - Each operations comes with a **fixed deposit** that is added to the **account** (defines the amortized cost).
  - For each operation, we subtract the cost of the operation from the account
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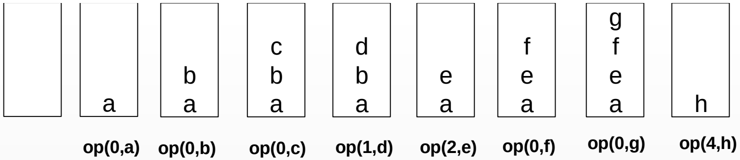
  - Iff the account is non-negative after each operation, the amortized cost is at most the fixed deposit.



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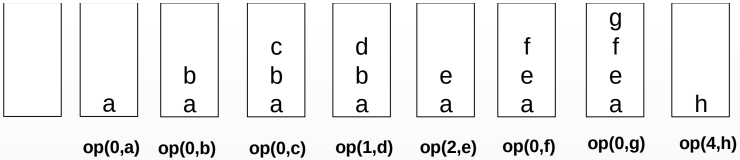
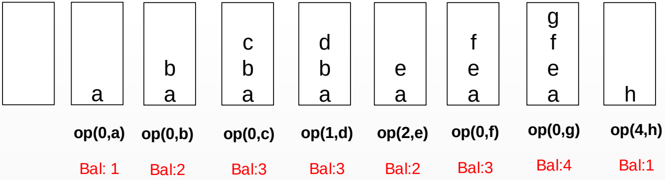
- Often, the account can be imagined as sum of 'credits' assigned to different components of data structure



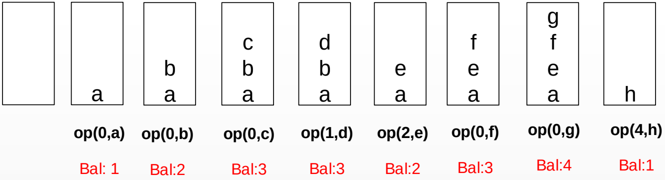
Accounting Method for Special Stacks

- We prove an amortized cost of 2 per operation $\rightarrow$ assume there is a fixed deposit of 2 per operation.



Accounting Method for Special Stacks

- We prove an amortized cost of 2 per operation → assume there is a fixed deposit of 2 per operation.
- Maintain this invariant: there is a credit of 1 for each item in the stack → account is the number of items in the stack.



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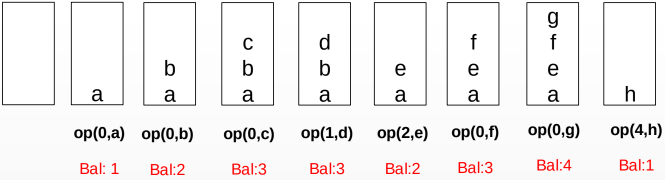

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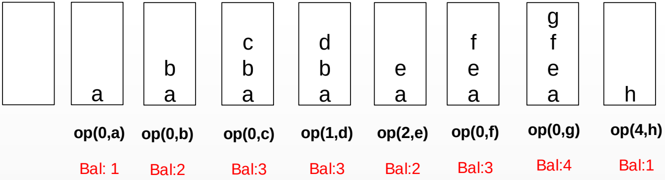

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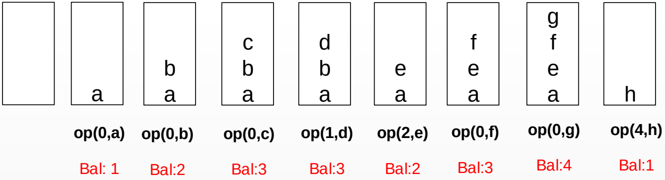
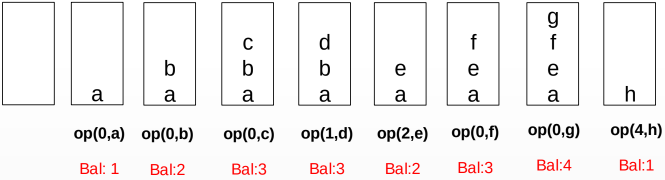
  - Push(x): there is a cost of 1 and fixed deposit of 2; the extra saving is stored as the credit for the item.



Accounting Method for Special Stacks

- With a fixed deposit of 2 per operation, we showed that the balance remains non-negative after each operation
- The balance was the accumulated credits stored in each item in the stack
- We conclude that the amortized cost of each operation is at most 2



Potential Method for Special Stacks

- Review: Define a potential function $\phi(i)$ which maps the state of the structure after operation i to a positive number.
  - Potential is equivalent to the available credit after each operation in the accounting method.
- Amortized cost is the summation of actual cost and the difference in potential function:
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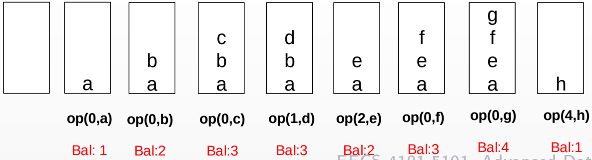

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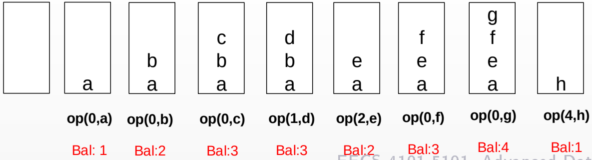

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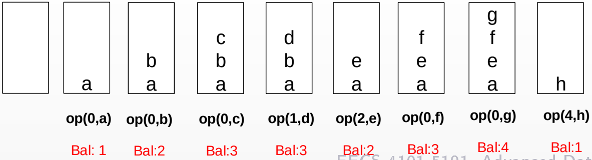

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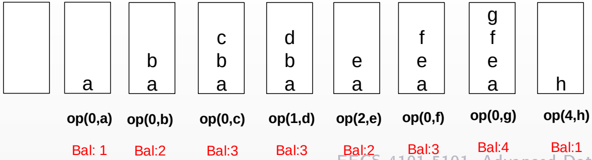

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- The whole field of online algorithms!