York University
LE/EECS 3101 A, Fall 2023
Assignment 1

Due Date: September 26th, at 23:59pm

The trees are about to show us how lovely it is to let the dead things go ...

anonymous

All problems are written problems; submit your solutions electronically **only via Crowdmark**. Some questions include an example and an answer that provides guidelines on how the solutions should look. Think of them as a tool for reviewing the material. Your solutions do not necessarily need to look like provided answers. You are welcome to discuss the general idea of the problems with other students. However, you must write your answers individually and mention your peers (with whom you discussed the problems) in your solution.

Throughout the assignment, all logarithms are based 2 logarithms, i.e., $\log x = \log_2(x)$.

Equations you must know:

- $1 + 2 + 3 + 4 + \ldots + i + \ldots + x = x(x+1)/2$
- $1 + 4 + 9 + \ldots + i^2 + \ldots + x^2 = x(x+1)(2x+1)/6$
- $1 + 2 + 4 + \ldots + 2^i + \ldots + 2^x = 2^{x+1} - 1.$
- $1 + 1/2 + 1/4 + \ldots \approx 2$
- $1 + 1/2 + 1/3 + \ldots + 1/n = \Theta(\log n)$

Problem 1 [4+4+4+4+4=20 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

Ex.) $15n^3 + 10n^2 + 20 \in O(n^3)$
Consider $M := 15 + 10 + 20 = 45$ and $n_0 := 1$. Then $15n^3 + 10n^2 + 20 \leq Mn^3$ for all $n \geq n_0$.

a) $n^2 + \frac{3n^2}{2+\cos(n)} \in O(n^2)$

b) $n^2(\log n)/10 \in \omega(n(\log n)^2)$.

c) $5n^2/(n+120) \in \Theta(n)$.

d) $1402n \in o(n \log n)$

e) $n^{2023} \in o(n^n)$

Problem 2 [4+4+4+4=16 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Provide a brief justification of your answers. In your justification you may use **any** relationship or technique that is described in class.

Ex.) $f(n) = n^{2.5}$ and $g(n) = n^2 \log(n)$.
We have $\lim_{n\to\infty} \frac{n^{2.5}}{n^2 \log n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \lim_{n\to\infty} \frac{(n^{-1/2})/2}{1/(n \ln 2)} = \lim_{n\to\infty} \frac{n^{1/2} \ln 2}{2} = \infty$. Hence we have $f(n) = \omega(g(n))$.

a) $f(n) = n(\log n)^3$ versus $g(n) = n^2$

b) $f(n) = \sqrt{n}$ versus $g(n) = (\log n)^4$

c) $f(n) = n^3(3 + 2\cos(2n^3))$ versus $g(n) = 2023n^3 + n^2 + 3n$

d) $f(n) = 4^n$ versus $g(n) = 3^{n/2}$

Problem 3 [5+5+5=15 marks]

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.

Ex.) $f(n) \in \Theta(g(n)) \Rightarrow g(n) \in \Theta(f(n))$
$f(n) \in \Theta(g(n))$, for large values of n we have $M_1 g(n) \leq f(n) \leq M_2 g(n)$ for some M_1 and M_2. This means we have $\frac{1}{M_2} f(n) \leq g(n) \leq \frac{1}{M_2} f(n)$, which shows $g(n) = \Theta(f(n))$.

a) $f(n) \notin o(g(n))$ and $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$

b) $f(n) \in \Theta(h(n))$ and $h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{g(n)} \in \Theta(1)$

c) $f(n) \in \Theta(g(n)) \Rightarrow 2^{f(n)} \in \Theta(2^{g(n)})$

Problem 4 [7 marks]

Analyze the following piece of pseudocode and give a tight (Θ) bound on the running time as a function of n. Show your work. A formal proof is not required, but you should justify your answer.

```
for i ← n^2 to 2n^2 do {
    rex ← 1
    for k ← 4i to 6i do
        rex ← rex * 2
    fido ← 1
    for k ← 0 to rex do
        fido ← rex * fido
}
```

Problem 5 5+5+5=15

Consider the following recursion (suppose n is a power of 2):

$$T(n) = \begin{cases} 4T(n/2) + n^{2.5} & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$

a) Use the alternation method to guess the asymptotic value of $T(n)$ (using Θ notation).

b) Use induction to prove the correctness of your guess in part (a) (it suffices to prove the value of $T(n)$ using O notation).

c) Draw the recursion tree for $T(n)$; specify the height of the tree, the number of leaves, and total work done in all levels of the tree. From your work, indicate the asymptotic value of $T(n)$.

Problem 6 [4+4+4+4=12 marks]

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply. For all cases, we have $T(x) = 1$ when $x \leq 100$ (base of recursion).

Ex.) $T(n) = 3T(n/3) + \sqrt{n}$
We have $n^{\log_b a} = n$. Since $f(n) = O(n^{1-\epsilon})$ (for any $\epsilon < 1/2$), we are at case 1 and $T(n) = \Theta(n)$.

a) $T(n) = 5T(n/3) + 2023n^{1.6}$

b) $T(n) = 9T(n/3) + 1984n^2$

c) $T(n) = 8T(n/2) + \frac{n^3}{\log n}$.

d) $T(n) = 16T(n/2) + n^4 \log^3 n$