York University
LE/EECS 3101 A, Fall 2023
Assignment 1

Due Date: September 26th, at 23:59pm

The trees are about to show us how lovely it is to let the dead things go ...

anonymous

All problems are written problems; submit your solutions electronically **only via Crowdmark**. Some questions include an example and an answer that provides guidelines on how the solutions should look. Think of them as a tool for reviewing the material. Your solutions do not necessarily need to look like provided answers. You are welcome to discuss the general idea of the problems with other students. However, you must write your answers individually and mention your peers (with whom you discussed the problems) in your solution.

Throughout the assignment, all logarithms are based 2 logarithms, i.e., $\log x = \log_2(x)$.

Equations you must know:

- $1 + 2 + 3 + 4 + \ldots + i + \ldots + x = x(x+1)/2$
- $1 + 4 + 9 + \ldots + i^2 + \ldots + x^2 = x(x+1)(2x+1)/6$
- $1 + 2 + 4 + \ldots + 2^i + \ldots + 2^x = 2^{x+1} - 1$.
- $1 + 1/2 + 1/4 + \ldots \approx 2$
- $1 + 1/2 + 1/3 + \ldots + 1/n = \Theta(\log n)$

Problem 1 [4+4+4+4+4=20 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

Ex.) $15n^3 + 10n^2 + 20 \in O(n^3)$
Consider $M := 15 + 10 + 20 = 45$ and $n_0 := 1$. Then $15n^3 + 10n^2 + 20 \leq Mn^3$ for all $n \geq n_0$.

a) $n^2 + \frac{3n^2}{2+\cos(n)} \in O(n^2)$
Answer: For any value of n, we have $\cos(n) \geq -1$ and hence $\frac{3n^2}{2+\cos(n)} \leq 3n^2$. Therefore, $f(n) \leq n^2 + 3n^2 = 4n^2$. So, it suffices to have $n_0 \geq 1$ and $M \geq 4$.

b) $n^2(\log n)/10 \in \omega(n(\log n)^2)$. **Answer:** We need to provide n_0 s.t. for all $M > 0$ and $n > n_0$ we have $n^2 \log n/10 > Mn(\log n)^2$, i.e., $n/\log n > 10M$. We know for $n > 16$, we have $\sqrt{n} > \log n$, which implies $n/\sqrt{n} < n/\log n$. So, we can write $n/\log n > n/\sqrt{n} = \sqrt{n}$. So, in order to have $n/\log n > 10M$, it suffices to have $\sqrt{n} > 10M$, which gives $n > 100M^2$. Therefore, it suffices choose $n_0 = \max\{100M^2, 16\}$.

c) $5n^2/(n+120) \in \Theta(n)$.
Answer: Suppose $n \geq 120$. Then we have $5n^2/(2n) \leq 5n^2/(n+120) \leq 5n^2/n$. That is, $2.5n \leq 5n^2/(n+120) \leq 5n$. Therefore, we can choose $n_0 = 120$ and $M_1 = 2.5$ and $M_2 = 5$.

d) $1402n \in o(n \log n)$
Answer: Given *any* value of M, we should provide n_0 so that $1402n < Mn \log n$, i.e,. $1402/M < \log n$. For this to hold, it suffices to have $n > 2^{1402/M}$. So it suffices to define n_0 as $\max\{1, 2^{1402/M}\}$.

e) $n^{2023} \in o(n^n)$
Answer: Set $n_0 := 2023 + M$. Then, for $n \geq n_0$, we have $n^n = n^{n-2023}n^{2023} \geq (2023+M)^M n^{20}$. Since $(2023+M)^M > M$, this shows that $0 \leq Mn^{2023} < n^n$ for all $n \geq n_0$.

Problem 2 [4+4+4+4=16 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Provide a brief justification of your answers. In your justification you may use **any** relationship or technique that is described in class.

Ex.) $f(n) = n^{2.5}$ and $g(n) = n^2 \log(n)$.
We have $\lim_{n\to\infty} \frac{n^{2.5}}{n^2 \log n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \lim_{n\to\infty} \frac{(n^{-1/2})/2}{1/(n \ln 2)} = \lim_{n\to\infty} \frac{n^{1/2} \ln 2}{2} = \infty$. Hence we have $f(n) = \omega(g(n))$.

a) $f(n) = n(\log n)^3$ versus $g(n) = n^2$ **Answer:** We have $\lim_{n\to\infty} \frac{n(\log n)^3}{n^2} = \lim_{n\to\infty} \frac{(\log n)^3}{n} = \lim_{n\to\infty} \frac{3(\log n)^2}{n \ln 2} = \lim_{n\to\infty} \frac{6 \log n}{n(\ln 2)^2} == \lim_{n\to\infty} \frac{6}{n(\ln 2)^3} = 0$. Hence we have $f(n) = o(g(n))$. Note that we applied L'Hopital rule three times.

b) $f(n) = \sqrt{n}$ versus $g(n) = (\log n)^4$ **Answer:** We can show $f(n) = \omega(g(n))$ by taking the limit and applying L'Hopital rule four times. Alternatively, we can use the definition as follows. We need to introduce n_0 so that for all $n > n_0$ and any M we have $n^{1/2} > M(\log n)^4$ which is equivalent to $(n^{1/8})^4 > (M^{1/4} \log n)^4$, i.e., it suffices to show $n^{1/8} > M^{1/4} \log n$. We note that for $n_0 > 2^{128}$ we have $n^{1/16} > \log n$ (to see that, note $2^{128/16} = 256 > 128$). So, in order to show $n^{1/8} > M^{1/4} \log n$, it suffices to show $n^{1/8} > M^{1/4} n^{1/16}$, which is equivalent to showing $n^{1/16} > M^{1/4}$. For this to hold, it suffice to have $n > (M^{1/4})^{16}$, i.e., define $n_0 = \max\{2^{128}, M^4\}$

c) $f(n) = n^3(3 + 2\cos(2n^3))$ versus $g(n) = 2023n^3 + n^2 + 3n$ **Answer:** For any value of n we have $2 \leq 3 + 2\cos(2n^3) \leq 3$. Hence $2n^3 \leq f(n) \leq 3n^3$. Similarly, for any value of n we have $2023n^3 \leq g(n) \leq 2027n^3$. We can conclude $f(n) \leq 3n^3 < 2023n^3 \leq g(n) \leq 2027n^3 < 1014 \times 2n^3 \leq 1014f(n)$. So, it suffices to have $M_1 \leq 1$, $M_2 \geq 1014$ and $n_0 = 1$.

d) $f(n) = 4^n$ versus $g(n) = 3^{n/2}$
Answer: We have $\lim_{n\to\infty} \frac{4^n}{3^{n/2}} = \lim_{n'\to\infty} \frac{4^{2n'}}{3^{n'}} = \lim_{n'\to\infty} \frac{16^{n'}}{3^{n'}} = \lim_{n'\to\infty} (16/3)^{n'} = \infty$. Note that $n' = n/2$. We conclude that $4^n \in \omega(3^{n/2})$.

Problem 3 [5+5+5=15 marks]

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.

Ex.) $f(n) \in \Theta(g(n)) \Rightarrow g(n) \in \Theta(f(n))$
$f(n) \in \Theta(g(n))$, for large values of n we have $M_1 g(n) \leq f(n) \leq M_2 g(n)$ for some M_1 and M_2. This means we have $\frac{1}{M_2} f(n) \leq g(n) \leq \frac{1}{M_2} f(n)$, which shows $g(n) = \Theta(f(n))$.

a) $f(n) \notin o(g(n))$ and $f(n) \notin \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n))$ **Answer:**
False. Counter example: Consider $f(n) := n$ and $g(n) := \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$. To prove the claim false it will be sufficient to show that $f(n) \notin O(g(n))$ and $f(n) \notin \Omega(g(n))$, since then the antecedent of the implication is satisfied while the consequent is not.

If $f(n) \in O(g(n))$, then there exist constants $n_0 > 0$ and $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq n_0$. But for any odd number $n_1 > c$ we have $f(n_1) = n_1 > c = cg(n_1)$, showing that $f(n) \notin O(g(n))$.

Similarly, if $f(n) \in \Omega(g(n))$, then there exists constants $n_0 > 0$ and $c > 0$ such that $cg(n) \leq f(n)$ for all $n \geq n_0$. But for any even number $n_1 > 1/c$ we have $cg(n_1) = cn_1^2 > n_1 = f(n_1)$, showing that $f(n) \notin \Omega(g(n))$.

b) $f(n) \in \Theta(h(n))$ and $h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{g(n)} \in \Theta(1)$
Answer: True. Proof: Let $n_1, n_2 > 0$ and $c_1, c_2, c_3, c_4 > 0$ be such that $c_1 h(n) \leq f(n) \leq c_2 h(n)$ for all $n \geq n_1$ and $c_3 g(n) \leq h(n) \leq c_4 g(n)$ for all $n \geq n_2$.

Since g and h are positive, for every $n \geq n_2$ we have

$$\frac{c_3}{h(n)} \leq \frac{1}{g(n)} \leq \frac{c_4}{h(n)}.$$

Let $n_0 = \max\{n_1, n_2\}$. Then for every $n \geq n_0$ we have

$$\frac{c_1 c_3 h(n)}{h(n)} \leq \frac{f(n)}{g(n)} \leq \frac{c_2 c_4 h(n)}{h(n)} \Rightarrow c_1 c_3 \leq \frac{f(n)}{g(n)} \leq c_2 c_4.$$

Selecting constants $c_1' = c_1 c_3$ and $c_2' = c_2 c_4$ we have $c_1' \leq \frac{f(n)}{g(n)} \leq c_2'$ for every $n \geq n_0$. Thus, according to the definition of Θ we have $\frac{f(n)}{g(n)} \in \Theta(1)$.

c) $f(n) \in \Theta(g(n)) \Rightarrow 2^{f(n)} \in \Theta(2^{g(n)})$ **Answer:** False. Counter example: Consider $f(n) = \log n$ and $g(n) = 2 \log n$. Then $f(n) \in \Theta(g(n))$ but $2^{f(n)} = n$ and $2^{g(n)} = n^2$.

Problem 4 [7 marks]

Analyze the following piece of pseudocode and give a tight (Θ) bound on the running time as a function of n. Show your work. A formal proof is not required, but you should justify your answer.

```
1.  for i ← n^2 to 2n^2 do {
2.      rex ← 1
3.      for k ← 4i to 6i do
4.          rex ← rex * 2
5.      fido ← 1
6.      for k ← 0 to rex do
7.          fido ← rex * fido
8.  }
```

Answer: The value of rex at the beginning of line 6 is 2^{2i}; this is because it starts at 1 and is doubled $6i - 4i = 2i$ times. So, the time complexity of lines 2 to 7 is $A(i) = c + \sum_{k=4i}^{6i} c' + \sum_{k=0}^{2^{2i}} c''$ for constant values of c, c', and c''. The value of $A(i)$ is thus $c + (6i - 4i + 1)c' + (2^{2i} + 1)c''$. The total time complexity of is thus

$$\begin{aligned} T(n) &= \sum_{i=n^2}^{2n^2} A(i) \\ &= \sum_{i=n^2}^{2n^2} c + (2i+1)c' + (2^{2i}+1)c'' \end{aligned}$$

The very last term in the sum is $(2^{4n^2} + 1)c'' \leq (2c'')2^{4n^2}$. Therefore, we can write $T(n) \geq (2c'')2^{4n^2}$.

Let $d = \max\{c, c', c''\}$. Then we can write $A(i) \leq d(2^{2i} + 2i + 3) < 3d2^{2i}$. The last inequality holds when $n \geq 2$ (and therefore $i \geq 2$). Therefore, we can write

$$T(n) \leq 3d \sum_{i=n^2}^{2n^2} 2^{2i} < 3d \sum_{i=1}^{2n^2} 2^{2i} < 3d \sum_{j=1}^{4n^2} 2^j = 3d(2^{4n^2+1} - 1) < 6d(2^{4n^2}).$$

The third inequality holds because on the left we have $3d(2^2 + 2^4 + \ldots + 2^{4n^2})$ and on the right we have $3d(2^1 + 2^2 + 2^3 + \ldots + 2^{4n^2})$.

To conclude, we have $(2c'')2^{4n^2} \leq T(n) \leq (6d)2^{4n^2}$, which certifies $T(n) = \Theta(2^{4n^2})$.

Problem 5 5+5+5=15

Consider the following recursion (suppose n is a power of 2):

$$T(n) = \begin{cases} 4T(n/2) + n^{2.5} & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$

a) Use the alternation method to guess the asymptotic value of $T(n)$ (using Θ notation).
Answer: We can write:

$$\begin{aligned} T(n) &= 4T(n/2) + n^{2.5} \\ &= 4(4T(n/4) + (n/2)^{2.5}) + n^{2.5} = 16T(n/4) + n^{2.5}(1 + 4/2^{2.5}) \\ &= 16(4T(n/8) + (n/4)^{2.5}) + n^2(1 + 4/2^{2.5}) = 64(n/8) + n^{2.5}(1 + 4/2^{2.5} + (4/2^{2.5})^2) \\ &= \ldots \\ &= 4^k T(n/2^k) + n^{2.5}(1 + (4/2^{2.5}) + (4/2^{2.5})^2 + \ldots + (4/2^{2.5})^{k-1}) \\ &= 4^{\log_4 n} T(1) + n^{2.5}(1 + (4/2^{2.5}) + (4/2^{2.5})^2 + \ldots + (4/2^{2.5})^{k-1}) \\ &= n^{\log_4 4} d + \Theta(n^{2.5}) \\ &= \Theta(n^{2.5}) \end{aligned}$$

b) Use induction to prove the correctness of your guess in part (a) (it suffices to prove the value of $T(n)$ using O notation).
Answer: For the based of induction, we have $T(2) = 4T(1) + 2^{2.5} = 4d + 2^{2.5} \leq M \cdot 2^{2.5}$, which holds as long as $M > 4d/2^{2.5} + 1$.
In the induction step, we can write

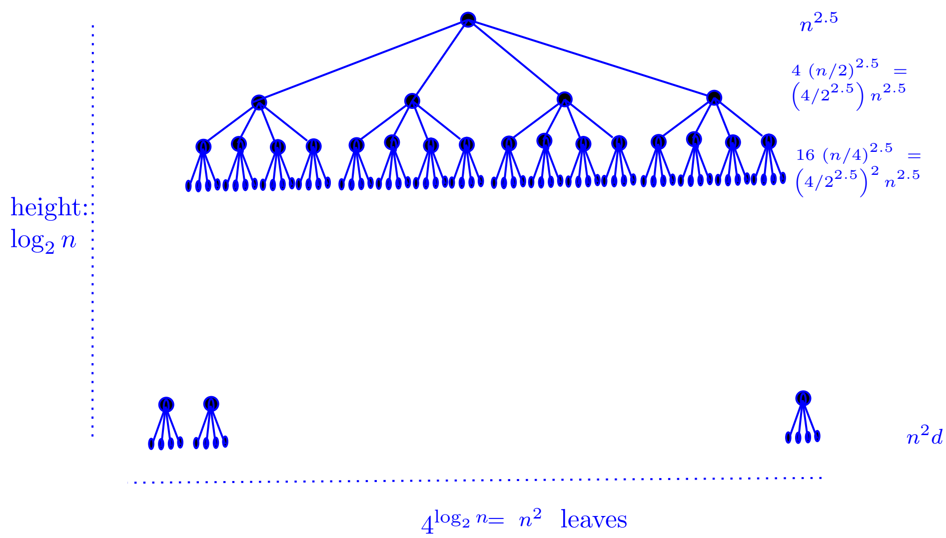
$$T(n) = 4T(n/2) + n^{2.5} \leq 4M.(n/2)^{2.5} + n^{2.5} = (4M/2^{2.5} + 1)n^{2.5}$$

So, it suffices to have $(4M/2^{2.5}+1)n^{2.5} \leq Mn^{2.5}$, or $M/\sqrt{2}+1 \leq M$, which holds as long as $M \geq 2/(\sqrt{2}-1)$. Therefore, it suffices to have $M = \max\{4d/2^{2.5} + 1, 2/(\sqrt{2}-1)$, and the desired inequality holds for all values of $n \geq 2$.

c) Draw the recursion tree for $T(n)$; specify the height of the tree, the number of leaves, and total work done in all levels of the tree. From your work, indicate the asymptotic value of $T(n)$.

Answer: The recursion tree is as follows; note the number of leaves and the height of the tree. The total work is (ignoring constants):

$$n^{2.5}(1 + (4/2^{2.5}) + (4/2^{2.5})^2 + (4/2^{2.5})^3 + \ldots + (4/2^{2.5})^{\log_3 n} + n^2 d) = \Theta(n^{2.5})$$



Problem 6 [4+4+4+4=12 marks]

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply. For all cases, we have $T(x) = 1$ when $x \leq 100$ (base of recursion).

Ex.) $T(n) = 3T(n/3) + \sqrt{n}$
We have $n^{\log_b a} = n$. Since $f(n) = O(n^{1-\epsilon})$ (for any $\epsilon < 1/2$), we are at case 1 and $T(n) = \Theta(n)$.

a) $T(n) = 5T(n/3) + 2023n^{1.6}$ **Answer:** We have $n^{\log_b a} = n^{log_3 5} = n^{1.4649}$ and hence $f(n) = \Theta(n^{1.6}) = \Omega(n^{\log_b a+\epsilon})$ for some ϵ (any $\epsilon < 0.13$ works). Hence we are at case III. For regularity condition, we have: $af(n/b) = 5(n/3)^{1.6} = 0.86n^{1.6} < cf(n)$ for $c \in (0.86, 1)$. Thus, regularity condition holds and we can state $T(n) = \Theta(n^{1.6})$.

b) $T(n) = 9T(n/3) + 1984n^2$
Answer: This is case 2 (for $k = 0$) and we can write $T(n) = \Theta(n^2 \log n)$.

c) $T(n) = 8T(n/2) + \frac{n^3}{\log n}$. **Answer:** We have $n^{\log_b a} = n^{\log_2 8} = n^3$. Note that $f(n) = \frac{n^3}{\log n} = O(n^3)$, but we cannot write $f(n) = O(n^{3-\epsilon})$ for any positive ϵ (there is no polynomial gap between f and $n^{\log_b a}$). Therefore we cannot apply Case 1 of Master theorem. Also note that $f(n) = \Theta(n^{\log_b a} \log^k n)$ for $k = -1$; since k is negative, we cannot apply Case 2 either. We conclude that Master theorem is not applicable in this example.

d) $T(n) = 16T(n/2) + n^4 \log^3 n$

Answer: We have $n^{\log_b a} = n^4$ and thus $f(n) = \Theta(n^{\log_a b} \log^k(n))$ for $k = 3$. Therefore, we are in Case 2 $T(n) = \Theta(f(n) \log n) = n^4 \log^4 n$.