LE/EECS 3101 A - Design & Analysis of Algorithms

Sample Midterm

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Write your name and student id here:

'All human beings are born free and equal in dignity and rights.' *Universal Declaration of Human Rights*

- **Do not open this booklet until instructed**.
- You are NOT allowed to use any printed/written material. A "cheat page" is provided at the end of the exam. It is OK to take the staples off, but make sure to **submit all sheets**, including the cheat sheet.
- Please **turn off your cell phones** and put them in your bags.
- Calculators are not needed but you can use simple calculators with no memory.
- Manage your time. We start the exam at 4:00 and end the exam at 5:30. **You have 90 minutes**. **Don't waste too much time on a single question. It is a long exam, and your time is limited.**
- To save trees, the exam is printed **double-sided**. There are **10 pages**, including this cover page, the cheat page, and two extra blank pages (use them if you need more space). You must submit ALL pages.
- If you find the exam too long/hard (which is likely), do not panic. The marks will be scaled so that the highest mark gets the full mark.

1. Short Answer Questions (24 marks)

In the actual midterm, this question will have 11 parts.

Provide your short answers in the provided boxes. There is no need to justify your answers.

1. True or False: adding two matrices A and B, each of size $n \times n$, takes $\Theta(n^3)$.

2. True or False: $n^2 \log n \in \Omega(n^3)$.

3. True or False: $n^3 \sin n \in \Theta(n^3)$

4. True or False: if $f(n) \in \Theta(g(n))$ then $f(n)^2 \in \Theta(g(n)^2)$

5. True or False: Strassen's Algorithm for matrix multiplication runs in $o(n^{2.5})$.

6. Assume $T(1) = 5$ and $T(n) = T(n/2) + n^2$. Give an expression for the run-time of $T(n)$ using Θ notation.

7. Assume $T(1) = 1401$ and $T(n) = 3T(n/3) + n$. Give an expression for the number of leaves in the recursion tree using Θ notation.

2. Asymptotic Analysis (4 marks)

Provide a complete proof of the following statement from first principles (i.e., using the original definitions of order notation).

$$n^{2.9} \in o(n^{4+\cos n})$$

3. Loop Analysis (5 marks)

Consider the following pseudocode:

```
foo(n)
    i ← 1
    prod ← 1
    while i < min{n, 2023} do
        for j = i to n do
            prod ← prod × j
        i ← i × 3
    return prod
```

What is the worst-case running time of *foo*(n)?
Express your answer using Θ-notation in terms of n, and be as precise as possible.
Show your work in the space below.

4. Divide & Conquer (10 marks)

Given an array A of integers with size n, devise a **divide and conquer** algorithm that reports the length of the longest contiguous subarray of A formed by even numbers. For example, for $A = [1, 0, 0, 3, 4, 4, 2, 2, 0, 0, 0, 3]$, the output should be 7, because there is a subarray formed by 7 even number.
You need to present the body of a procedure with header $\text{LongestEvenSubArray}(A, lo, hi)$ (you are expected to know the meaning of lo and hi). The first call is $\text{LongestEvenSubArray}(A, 0, n-1)$.

5. Heap Operations (6 marks)

Consider array $A = [6, 0, 8, 12, 1, 13, 3, 11, 2, 7, 9, 10, 5, 4, 15]$.

a) Apply the Heapify procedure on A. Show the resulting tree in the space below.

b) On the original heap, apply operation *insert(14)*. Show the resulting tree in the space below.

6. Median-of-Five Variant (8 marks)

a) Consider a variant of Median-of-Five algorithm in which, instead of partitioning input into $n/5$ blocks of size 5, we partition the input into $n/7$ blocks of size 7.

Follow the same steps as in the slides to derive a recursive formula for the time complexity $T(n)$ of this algorithm (there is no need to solve the recursion to find the time complexity of this variant of the algorithm).

b) Try to solve the recursion by guessing that $T(n) \in O(n)$. Follow the same steps as in the slides and indicate whether we can state $T(n) \in O(n)$.

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Remember to submit this page.

$$x^a x^b = x^{a+b}$$
$$(x^a)^b = x^{ab}$$
$$x^n + x^n = 2x^n \neq x^{2n}$$
$$x^n * x^n = x^{2n}$$
$$x^0 = 1$$
$$x^{-a} = \frac{1}{x^a}$$
$$\frac{x^a}{x^b} = x^{a-b}$$
$$(xy)^a = x^a y^a$$
$$2^n + 2^n = 2^{n+1}$$

$$\log_b(M) = X \leftrightarrow b^X = M$$
$$\log_b(b) = 1$$
$$\log_b(1) = 0$$
$$\log_b(0) \rightarrow \text{not defined}$$
$$\log_b(b^k) = k$$
$$b\log_b(M) = M$$
$$\text{if} \log_b(M) = \log_b(N), \text{ then } M = N$$
$$\log_b(M^k) = k \cdot \log_b(M)$$
$$b^{\log_b(M)} = M$$
$$\log_b\left(\frac{1}{M}\right) = \log_b(M^{-1}) = -1 \cdot \log_b(M)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M \cdot N) = \log_b(M) + \log_b(N)$$
$$\log_b(M) = \frac{\log_c(M)}{\log_c(b)}$$

$$a^{\log_b n} = n^{\log_b a}$$

$$\sum_{i=1}^{n} 1 = n$$
$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$$
$$\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k = \frac{1}{k+1}n^{k+1}$$
$$\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1}-1}{a-1}(a \neq 1); \quad \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$
$$\sum_{i=1}^{n} i2^i = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$$
$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma, \text{ where } \gamma \approx 0.5772\ldots \text{ (Euler's Constant)}$$
$$\sum_{i=1}^{n} \lg i \approx n \lg n$$

$$T(n) = \begin{cases} a\,T\left(\frac{n}{b}\right) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

($a \geq 1$, $b > 1$, and $f(n) > 0$)

- Compare $f(n)$ and $n^{\log_b a}$
- Case 1: if $f(n) \in O(n^{\log_b a - \epsilon})$, then $T(n) \in \Theta(n^{\log_b a})$
- Case 2: if $f(n) \in \Theta(n^{\log_b a}(\log n)^k)$ for some non-negative k then $T(n) \in \Theta(f(n)\log n) = \Theta(n^{\log_b a}(\log n)^{k+1})$
- Case 3: if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ and if $af(n/b) \leq cf(n)$ for **some constant** $c < 1$ (regularity condition), then $T(n) \in \Theta(f(n))$

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