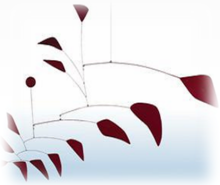

# EECS 3101 - Design and Analysis of Algorithms

**Shahin Kamali**

Topic 7 - Remarks on Complexity

# Overview

- A few remarks about complexity classes
- A brief introduction to NP-completeness
- $P \neq NP$ and other questions for which we do not know the answers!

# Polynomial Algorithms

- Most algorithms you have seen have running times $\Theta(\log n)$ (e.g., binary search), $\Theta(n)$ (e.g., searching in a linked list), $\Theta(n \log n)$ (e.g., merge-sot), $\Theta(n^2)$ (e.g., bubble-sort), $\Theta(n^3)$ (e.g., matrix multiplication), etc.
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  - Whether a polynomial time algorithm exists was not-known until 2002.
  - Three scientists from India discovered an algorithm (later named AKS primality test) that runs in $\tilde{O}((\log x)^6) \in O((\log x)^7)$.
  - Their paper won the 2006 Godel prize among other things.


Manindra Agrawal


Neeraj Kayal


Nitin Saxena

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  - Modern 3Sum-conjecture: 3-Sum requires $\Omega(n^{2-\epsilon})$ time for any constant $\epsilon > 0$.
  - If this conjecture is true, many other **3SUM-hard** problems also requires $\Omega(n^{2-\epsilon})$.

# Exhaustive Search

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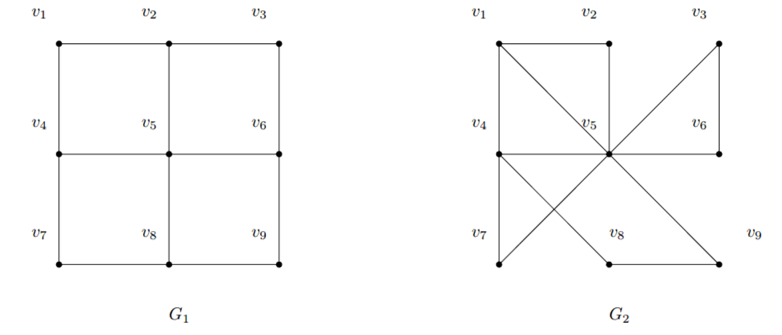
  - This is not good!
  - For many problems, we have failed to do much better.

# Hamiltonian Path

- Instance: a graph $G$ with vertex set $V$ and edge set $E$.
- Question: Does there exist a path in $G$ that visits every vertex in $V(G)$ exactly once along a sequence of edges in $E(G)$?



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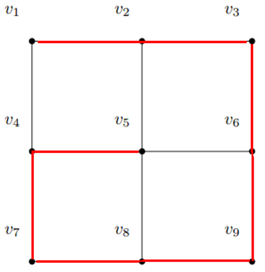
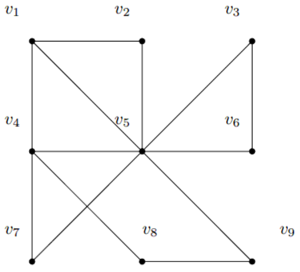



$G_1$




$G_2$

# Exhaustive Search for HP

- Try all paths and check whether the sequence of edges exist in $G$
- In other words, try all permutations of vertices
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  - Regardless, the number of distinct paths is still $\Theta(n!)$.
- $\rightarrow$ exhaustive search requires $\Omega(n!)$ in the worst case

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- There are many **'Hard'** problems like Hamiltonian path problem for which we do not know whether a polynomial algorithm exists; they form a complexity class.
  - If there is a polynomial algorithm for any of these problems, there will be polynomial algorithms for all of them.
  - When you fail to come up with a polynomial algorithm for a problem, investigate whether it is 'Hard'.

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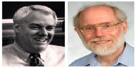
- Assume you have a problem $P$ for which you look for an efficient, polynomial algorithm, and you fail after trying a bit.
- How can you determine whether you should keep searching for an efficient algorithm or whether it's unlikely that any efficient algorithm for problem $P$ exists?
- If you can reduce one of those Hard problems to $P$ in polynomial time, then a polynomial algorithm for $P$ gives polynomial algorithms for all those hard problems.

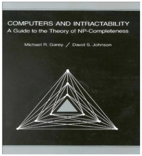
# Application of Reductions

- Since none of those Hard problems have any known polynomial algorithm, it is unlikely that you can come up with a polynomial algorithm for $P$.
    - Informally, to give up searching for a polynomial algorithm for $P$, it suffices to reduce a 'Hard' problem to $P$ in polynomial time.
    - We say the problem is **NP-Hard** in that case!
    - To show $P$ is NP-Hard, we reduce another NP-Hard problem to $P$

"I can't find an efficient algorithm, but neither can all these famous people."



Michael Garey David Johnson



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- Important complexity classes: P, NP, EXP, R, etc.
- P = problems that can be solved in polynomial time, i.e., $O(n^c)$ for some fixed $c$
  - E.g., given a graph on $n$ vertices and $m$ edges, find its MST; it can be done in $O(n^3)$.
  - Basically, all problems for which you have seen an algorithm belong to class $P$ of problems.

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- These are problems whose solutions can be verified in polynomial time.
    - For decision problems, instances with a **yes** answer can be verified.
- E.g., Hamiltonian Path is an NP problem: given an instance of the problem we can verify if a solution gives a 'yes' answer in polynomial time.
    - Given a solution path, we can verify whether it is a Hamiltonian path, i.e., check whether it visits every vertex exactly once, in polynomial time (in $O(n \log n)$ exactly).

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  - Yes, given a solution (3 numbers from the set), we can verify in polynomial time whether they sum to 0.

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  - To date only one of the Millennium has been solved, the Poincare Conjecture, solved by Perelman in 2006; he declined the money. He was also awarded Fields medal and rejected it: *"I'm not interested in money or fame; I don't want to be on display like an animal in a zoo"*.

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  - If a problem can be solved in polynomial time, its solutions can be checked in polynomial time as well, i.e., $P$ is a subset of NP.
  - The other direction is conjectured to be false, i.e., it is conjectured that there are problems which are in NP but not P, i.e., no polynomial algorithm exists for them.
  - Recall this problem ($NP \in P$) which is equal to $P = NP$ is open.

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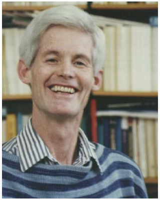

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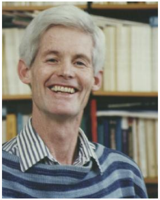

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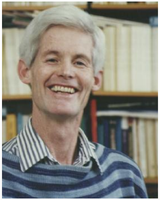

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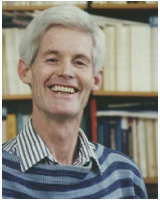
  - defined the concepts of reduction, NP-hardness, and NP-completeness
  - showed that every problem in NP reduces to boolean satisfiability problem (SAT) $\rightarrow$ SAT is NP-hard.



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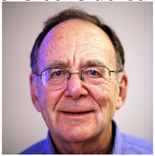
  - Karp got his Turing award in 1986; partially because his contribution to complexity theory.



# NP-hard problem Consequences

- If a problem $Q$ is NP-hard:
  - All NP-problems reduce to A in polynomial time, i.e., it is at least as hard as any NP problem.
  - Upper bound consequence: if we have a polynomial algorithm that solves $Q$, then there will be polynomial algorithms for all NP problems.
  - Lower bound consequence: if we show there is no polynomial algorithm for any NP problem, then there is no polynomial algorithm for $Q$.

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- Note that there are NP-problems which are not NP-complete (e.g., 3Sum or MST) and there are NP-hard problems that we do not know whether they belong to NP (EXP-complete problems).

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- If we show a problem is NP-complete, we often stop any effort for designing any polynomial algorithm or devising a polynomial time lower bound (just give up on finding exact solutions for the problem).
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- Steps for showing NP-completeness of a problem $A$:
  - Show $A$ is in NP, i.e., show that a yes instance of size $n$ can be verified in polynomial time (i.e., $O(n^c)$).
  - Show that $A$ is NP-hard, i.e., prove that all NP problem reduce to A in polynomial time

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  - Show that the reduction can be computed in time $O(n^c)$ (polynomial time).

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  - An upper bound $O(f'(n))$ for $H$ applies to $E$, assuming $f'(n)$ is not dominated by the reduction time.
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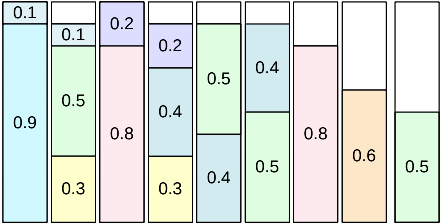
- An upper bound for $Y$ also applies to $X$, i.e., in particular if there is a polynomial time algorithm for $Y$, then that algorithm can be used to answer $X$ (and all NP problems which reduce to $X$) in polynomial time. This implies that $P = NP$.

# Bin Packing Problem

- The input is a **multi-set** of items of various sizes in range (0,1].
- The goal is to pack these items into a minimum number of bins of uniform capacity.
  - E.g., $S = \{0.1, 0.2, 0.2, 0.3, 0.3, 0.4, 0.4, 0.4, 0.5, 0.5, 0.5, 0.5, 0.6, 0.8, 0.8, 0.9\}$



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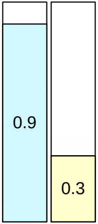

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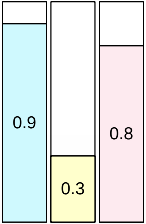

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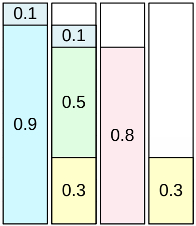

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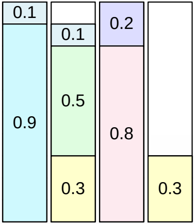

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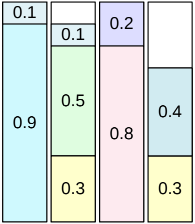

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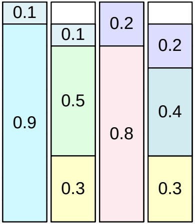

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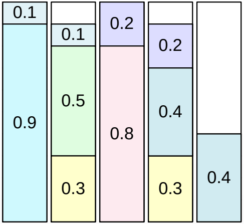

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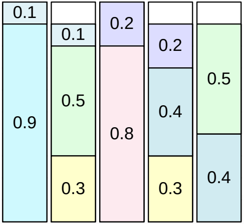

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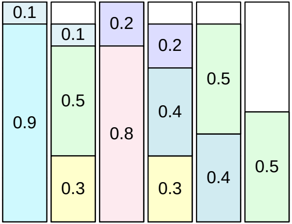

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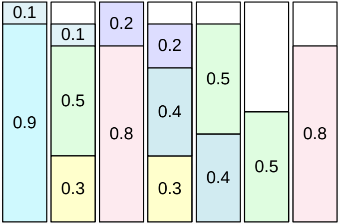

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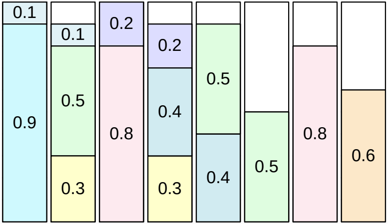

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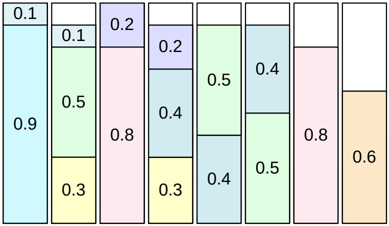

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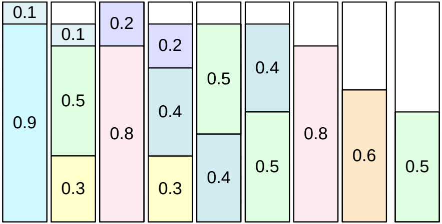

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- Server consolidation (e.g., in cloud)
  - Servers are bins and items are clients (e.g., cloud tenants) and you want to minimize the number of active servers.

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  - If all numbers are $O(n^c)$, there is an algorithm that runs in polynomial time; that is called a **pseudo-polynomial** algorithm.

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  - Note that we have $\sum_{q_i \in Q} q_i = \frac{2}{t} \cdot \sum_{p_i \in P} p_i = \frac{2}{t} \cdot t = 2$.

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- **Show that the answer to the partition instance $P = \{p_1, p_2, \ldots p_n\}$ is yes if and only the answer to in packing instance $Q = \{q_1, \ldots, q_n\}$ is yes (i.e., items can be packed in 2 bins)**.
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- We show that the bin packing instance can be packed into 2 bins.
  - Since $\sum_{p_i \in S} p_i = \sum_{p_i \in P-S} p_i = t/2$, we have $\sum_{p_i \in S} q_i = \sum_{p_i \in P-S} q_i = \frac{t}{2} \cdot \frac{2}{t} = 1$.
  - We can pack the items associated with set $S$ (i.e., set of $q_i$'s s.t. $p_i \in S$) in one bin and the rest in another.
  - The total size in each bin will not be more than 1 (hence a valid packing).

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  - We have $\sum_{p_i \in S} p_i = \sum_{p_i \in S} q_i \cdot \frac{t}{2} = \frac{t}{2}$.
  - So, $S$ and $P - S$ will be two subsets of the partition instance each with total sum of $t/2 \rightarrow$ the answer to partition instance is yes.

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**Bin Packing is an NP-complete problem.**

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# The Ending

**Observation**

*You should aim for the stars - and hopefully avoid ending up in the clouds!* Roxanne McKee

- We covered some materials about algorithms & complexity; the goal was not to cover everything; but prepare you to get interested and discover yourself in your future career.
- When dealing with a problem, we are interested in:
  - designing algorithms for them (using tools such as data structures)
  - analyzing algorithms (based on time complexity, memory requirement, approximation ratio, etc.) to provide guarantees.
  - understanding the restrictions of algorithms (lower bounds and complexity classes).
- 99 percent of people who talk about algorithms (e.g., in media, news, etc.) don't understand them. Hopefully you are not one of them any more.

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- Your feedback is appreciated; if something can be improved (which is 100 percent the case), let me know.
- I hope to see you in future courses.