Inapproximability for planar embedding problems

Jeff Edmonds[1] Anastasios Sidiropoulos[2] Anastasios Zouzias[3]

[1]York University

[2]TTI-Chicago

[3]University of Toronto

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Metric Space

$M = (X, D)$ is a metric space.

- X is a set.
- D is a distance function on X, i.e., satisfies triangle inequality.

Examples

- Any normed space.
- Graphs with shortest path distance.
- $\ldots$
- $\ldots\ \ldots$

Embedding between metric spaces

- Given $M = (X, d_X)$ and $M' = (Y, d_Y)$.
- Embedding $f : X \mapsto Y$.

Metric distortion

f has *distortion* α if

$$\forall x_1, x_2 \in X \quad d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \alpha \cdot d_X(x_1, x_2).$$

- $\mathrm{dist}(f) = \max \exp(f) \times \max \mathrm{contr}(f)$
- Well-studied subject: *Worst case* distortion.
- Relative Embeddings: Given X, find near-optimal embedding $f : X \mapsto Y$ efficiently.

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Computational problems (Approximate min. distortion)

Bijection and Injection

Bijection

Given two finite metric spaces X, Y of the same size n. Approximate the minimum distortion bijection $f : X \mapsto Y$.

Introduced in [KRS04].

Injection

Given finite metric space X of size n and infinite metric space Y with fixed dimensionality. Approximate the minimum distortion injection of $f : X \mapsto Y$.

Remark: Although different problems, share the same approximability.

Notation α vs. β: Given X it is NP-hard to check if $\exists\, f : X \mapsto Y$ with distortion $\leq \alpha$ or every f has distortion $> \beta$. Notice that $1 \leq \alpha < \beta$.

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Related Work

Dimension	Approximability		References
	Bijection ($X, Y \subseteq \mathbb{R}^d$)	Injection ($Y = \mathbb{R}^d$)	
$d = 1$	*OPT*, if dist $\leq 3 + 2\sqrt{2}$		[KRS04]
	OPT, if dist $\leq 5 + 2\sqrt{6}$		[CMO+08]
	$\mathrm{poly(n)}$ vs. $\mathrm{poly(n)}$	$\mathrm{poly(n)}$ vs. $\mathrm{poly(n)}$	[HP05], [BCIS05]
$d = 2$		NP-hard	[BCIS06]
	c_1 vs. c_2	c'_1 vs. c'_2	This paper
		$\mathrm{poly(n)}$ vs. $\mathrm{poly(n)}$	[MS08]
$d \geq 3$	a vs. $3a$	NP-hard	[PS05], [Edm07]
	a vs. $\Omega(\log^{1/4-\varepsilon} n)a$	c vs. $\mathrm{poly(n)}$	[KS07], [MS08]

Notation α vs. β: Given X it is NP-hard to check if $\exists f : X \mapsto Y$ with distortion $\leq \alpha$ or every f has distortion $> \beta$. Notice that $1 \leq \alpha < \beta$.

Our Results

It is NP-hard to decide whether the minimum distortion of

1. a *bijection* between two finite subsets of $\mathbb{R}^2$ under ℓ_2 is at least α or at most β, where $1 < \alpha < \beta$.
2. an *injection* of a finite metric space onto $\mathbb{R}^2$ under ℓ_∞ is at least α' or at most β', where $1 < \alpha' < \beta'$.

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Bijection Proof Outline

Outline

1. Given 3SAT formula ϕ. Construct instance of bijection problem.
2. Construct pair $X, Y \subseteq \mathbb{R}^2$, $|X| = |Y|$ s.t.
   - If ϕ is SAT, then X embeds into Y with distortion at most α.
   - If $f : X \mapsto Y$ bijection with distortion at most β, then ϕ is SAT.

Key Ideas:

- Locally there are *two* possible low-distortion bijections between $X \to Y$. *Encode* binary decision.
- Bypass *crossing* obstacle (as in [PS05, KS07]) by considering different scales when crossing.

Description of construction: By giving subsets of input ($\bullet \in X$) and target space ($\circ \in Y$) *simultaneously*.

$$f(\bullet) \to \circ$$

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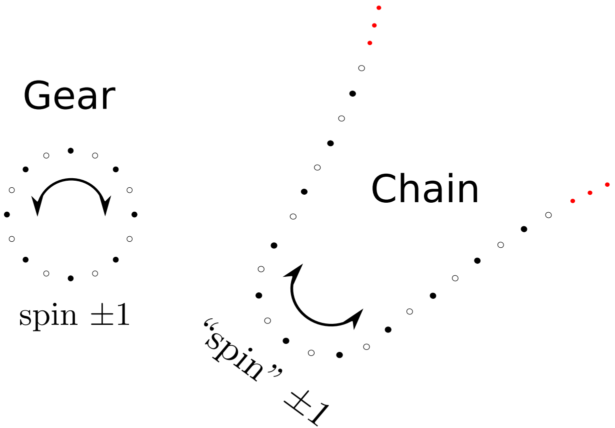
The construction

Gears - Chains

Reminder
$f(\bullet) \rightarrow \circ$



Main Idea: Sufficient low-distortion $\Longrightarrow$ gears spin and chains "spin".

The construction - Details

Gear

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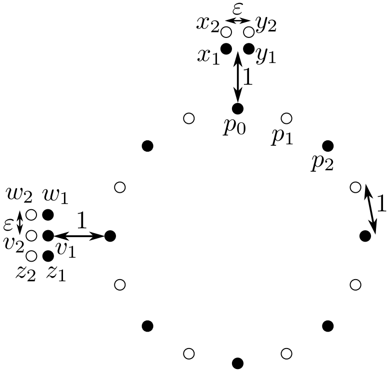


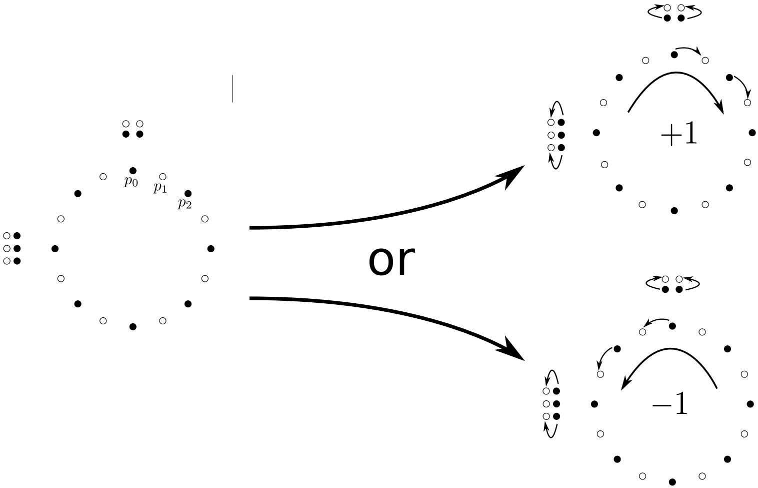
- Chain is similar but open.

The construction - Details

Binary Decision

Reminder

$f(\bullet) \to \circ$



Main Idea: In any low-distortion f only two embeddings, i.e., spin clock-wise or counter-clockwise.

The construction - Details

Connecting Gear/Chain

Reminder
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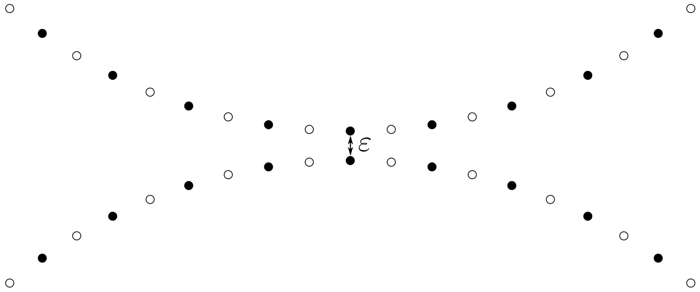

Key point Sufficient low-distortion $\implies$ neighbor gears and gears/chains have opposite spins.

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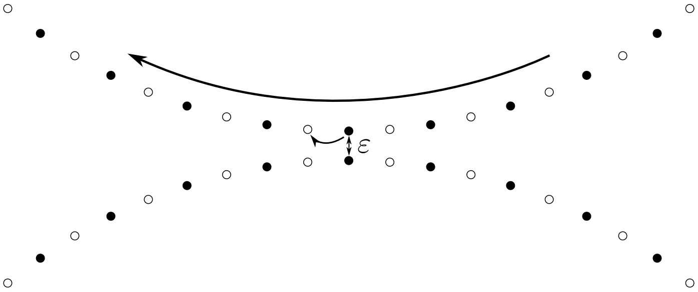

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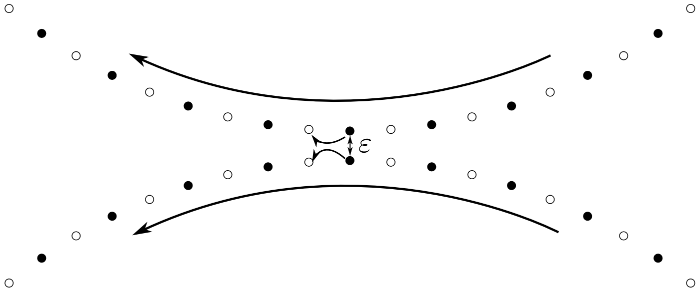

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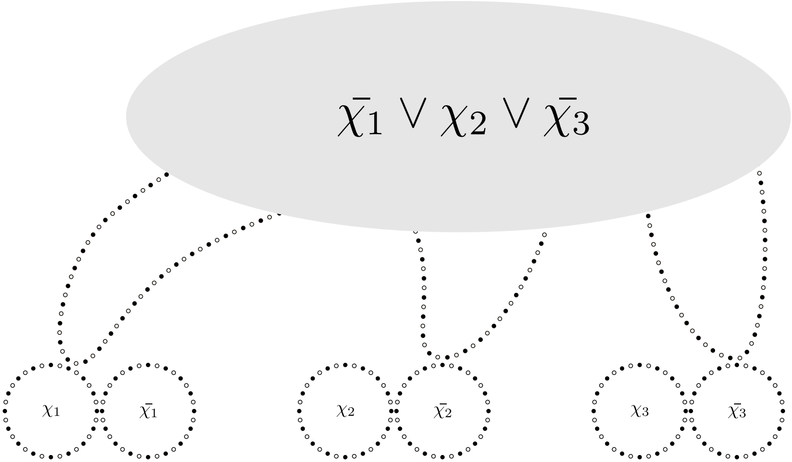
The construction

Connection - Clause

Reminder

$f(\bullet) \to \circ$



Key point Sufficient low-distortion $\implies$ opposite spins.

Clause Connect chains to encode a boolean constraint.

The construction
1-in-3 3SAT

Reminder
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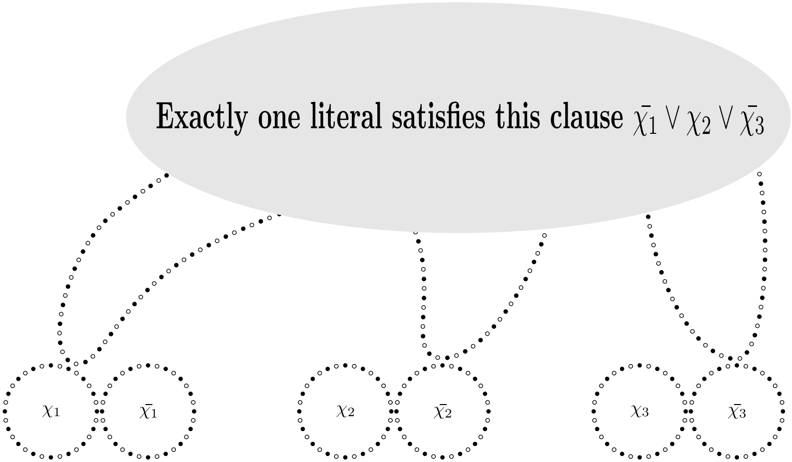
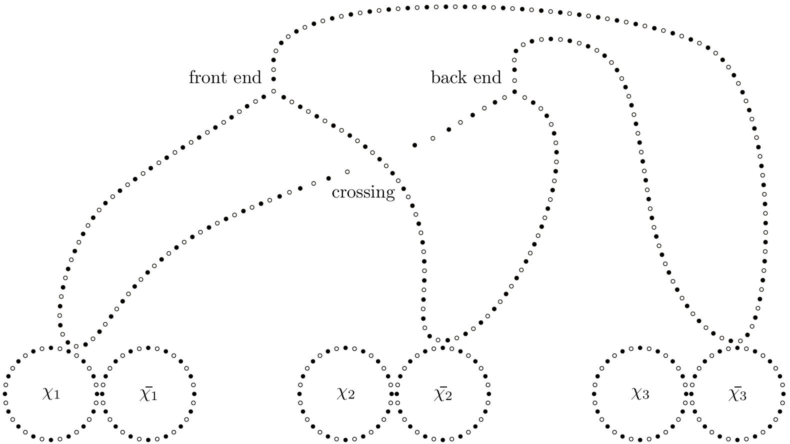


- Restrict each clause to be satisfied by exactly one literal. 1-in-3 3SAT is NP-complete [Sch78].

Final construction

Reminder
$f(\bullet) \to \circ$

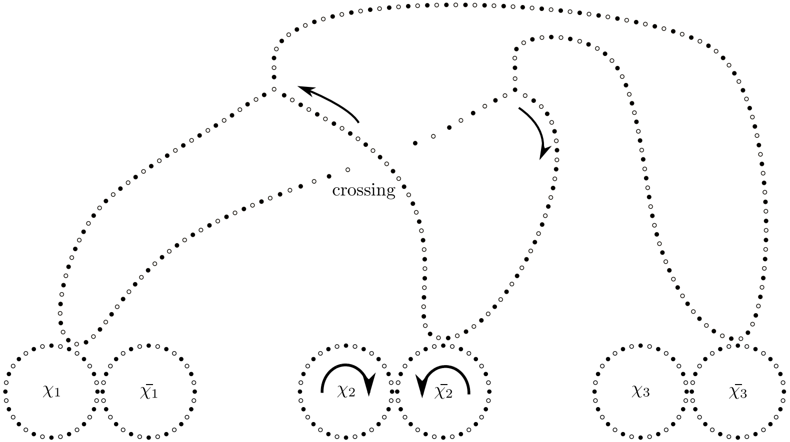


- Notice that $|X| = |Y|$.
- This subset of X, Y encodes the 1-in-3 clause $\bar{\chi_1} \vee \chi_2 \vee \bar{\chi_3}$.

Final construction

1-in-3 sat spin

Reminder
$f(\bullet) \rightarrow \circ$



true

- A 1-in-3 SAT assignment of $\bar{\chi}_1 \vee \chi_2 \vee \bar{\chi}_3$.

Final construction

1-in-3 sat spin

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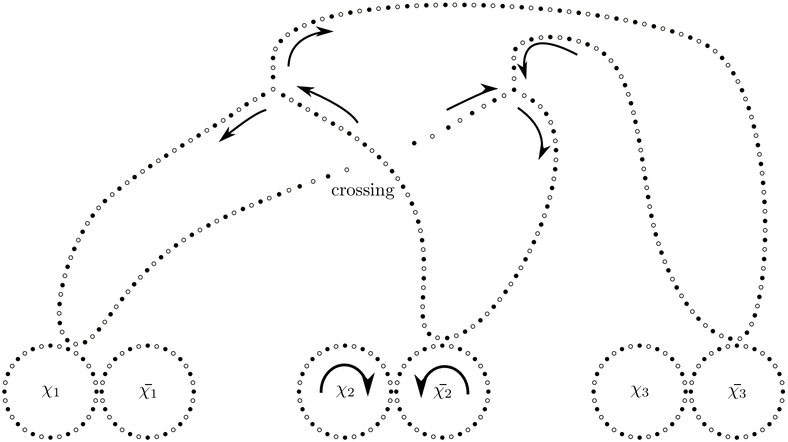

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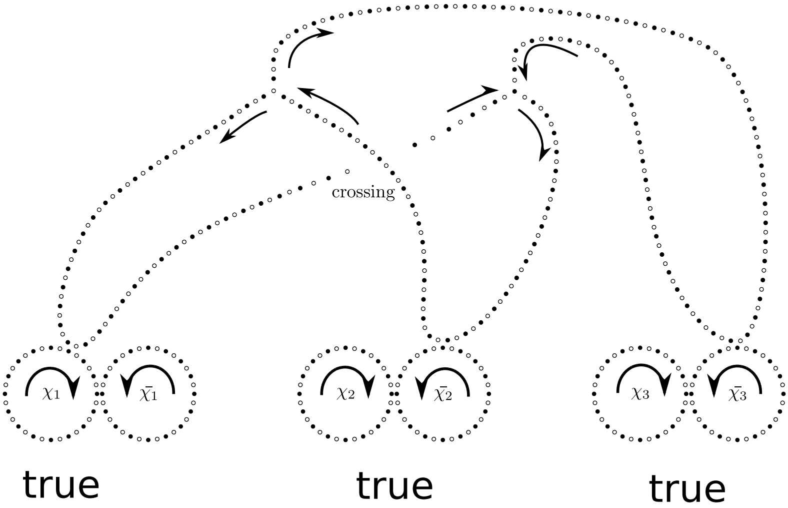

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The construction - Details

How to deal with crossings?

Reminder

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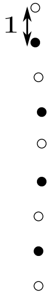

- Vertical and horizontal chains.

The construction - Details

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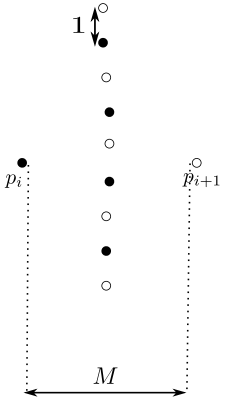

- Vertical and horizontal chains.
- Gap $M \implies$ No vertical point is mapped to horizontal chain.

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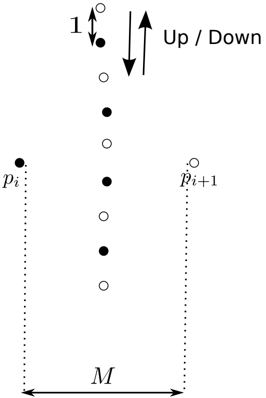

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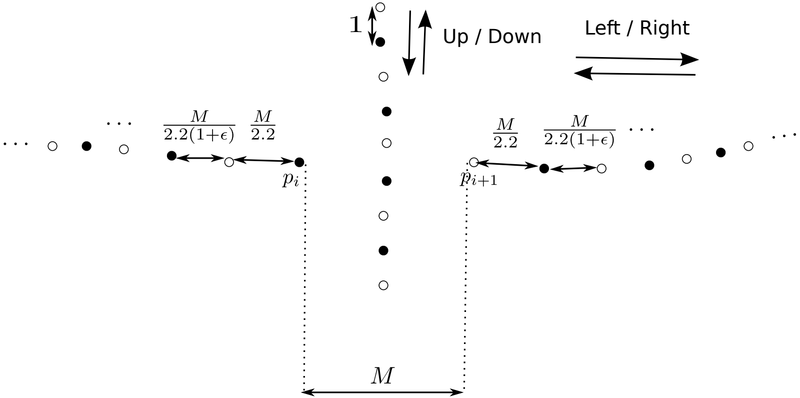

- Vertical and horizontal chains.
- Gap $M \implies$ No vertical point is mapped to horizontal chain.
- Horizontal chain's distances change exponentially.

Analysis

We show the following:

Yes instances

- If ϕ is 1-in-3 3SAT, then exists f with distortion at most α.
- Simple calculations give that $\alpha = 3.61 + \varepsilon$.

No instances

- For any f with distortion $\leq \beta$, we construct a 1-in-3 sat assignment for ϕ.
- If the distortion is at most $\beta = 4 - O(\varepsilon)$. Then
  - The spins are still well-defined
  - Neighborly gear/chains have opposite spin
  - Hence an 1-in-3 assignment for ϕ if well-defined

Summary

- Inapproximability results for planar bijection problem.
- Inapproximability for injection problem requires significantly different ideas.

Open Problems:

- Tighten the approximation gap, i.e., values of best constants α and β.
- Approximability when distortion $\approx 1+\varepsilon$.
- Is there an efficient algorithm when the optimal distortion is at most $1+\varepsilon$ for $\mathbb{R}^2$, similar to [KRS04, CMO$^+$08].

Thank You

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