Balanced Allocations of Cake

Jeff Edmonds [*] *Kirk Pruhs* [†]

Abstract

We give a randomized algorithm for the well known caking cutting problem that achieves approximate fairness, and has complexity $O(n)$. The heart of this this result involves extending the standard offline multiple-choice balls and bins analysis to the case where the underlying resources/bins/machines have different utilities to different players/balls/jobs.

[*] York University, Canada. jeff@cs.yorku.ca. Supported in part by NSERC Canada.

[†] Computer Science Department. University of Pittsburgh. kirk@cs.pitt.edu. Supported in part by NSF grants CCR-0098752, ANI-0123705, CNS-0325353, and CCF-0448196, and CCF-0514058.

1 Introduction

The protocol's goal in the well known cake cutting problem is to fairly apportion some resources among n players. Here we consider a continuous resource, modeled, without great loss of generality, by the unit interval. We assume that each player p has an initially unknown value function V_p that specifies how player p values each subinterval of the unit interval. A portion is a union of disjoint subintervals, and the value function is additive, so that the value of a portion is the sum of the values of the underlying subintervals. A player believes that a portion is c-fair if that portion has value at least $\frac{c}{n}$ of the total value of cake according to his value function. In the standard model, the protocol is allowed to make two types of queries to the players. In an evaluation query, the protocol asks a player how much he values a particular subinterval of the cake. In a cut query, the protocol asks the player to identify the shortest subinterval with a fixed value and a fixed left endpoint. We are interested in the query complexity of a protocol, which is the worst-case number of queries required to achieve a fair allocation for each player that follows the protocol.

The cake cutting problem originated in 1940's Polish mathematics community. Since then the problem has blossomed and been widely popularized. The motivation for using cake as a resource is the well known phenomenon that some people prefer frosting, while others do not. Cake cutting, and related fair allocation problems, are of wide interest in both social sciences and mathematical sciences. Sgall and Woeginger [11] provide a nice brief overview. There are several books written on fair allocation problems, such as cake cutting, that give more extensive overviews, see for example [3, 10]. Some quick Googling reveals that cake cutting algorithms, and their analysis, are commonly covered by computer scientists in their algorithms and discrete mathematics courses.

A deterministic 1-fair protocol with complexity $\Theta(n^2)$ was described in 1948 by Steinhaus in [12]. In 1984, Evan and Paz [5] gave a deterministic divide and conquer 1-fair protocol that has complexity $\Theta(n \log n)$. Recently, there has been several lower bound results for cake cutting. In particular, we showed that the Even-Paz algorithm is optimal for deterministic 1-fair protocols [4]. That is, every deterministic 1-fair protocol for cake cutting has complexity $\Omega(n \log n)$. This lower bound also applies to deterministic protocols that need only only guarantee $O(1)$-fairness. Sgall and Woeginger [11] showed that every randomized 1-fair protocol has complexity $\Omega(n \log n)$ if every portion is restricted to be a contiguous subinterval of the cake. We showed that every randomized $O(1)$-fair protocol has complexity $\Omega(n \log n)$ if there is a small relative error in the response to the queries [4].

A natural open question is then whether there exists a protocol with linear complexity for the any of the variants of cake cutting considered in the literature. In this paper we answer this question in the affirmative. Our protocol is randomized, requires exact answers to the queries, guarantees only $O(1)$-fairness, and does not in general assign a contiguous subinterval to each player. That is, we show that linear complexity is obtainable in the variant that is most in the protocol's favor. Additionally, we show that $O(n)$-complexity is still achievable even if there is a small relative error in the response to the queries, as long as the error that results from a cut query is independent of value in the query. We call this a weak adversary. All of the known results are summarized in Table 1.

The heart of our cake cutting algorithm is the following Balanced Allocation Lemma in the cake model that generalizes the standard multiple-choice balls and bins model [8].

Lemma 1 (Balanced Allocation). *Let $\alpha \geq 10$ be some sufficiently large constant. Each of n players has a partition of the unit interval $[0, 1]$, or cake, into αn disjoint candidate subintervals/pieces. Each player independent picks $d' = 2d = 4$ of his pieces uniformly at random, with replacement. Then there is an efficient method that, with probability $\Omega(1)$, picks one of the d' pieces for each player, so that every point on the unit interval is covered by $O(1)$ pieces.*

Deterministic vs. Randomized Protocol	Exact vs. Approximate Queries	Standard vs. Weak Adversary	Exact vs. Approximate Fairness	Contiguous vs. Non-contiguous Portions	Complexity	Reference
*	Exact		*	*	$O(n \log n)$	[5]
*	*		Exact	Contiguous	$\Omega(n \log n)$	[11]
Deterministic	*		*	*	$\Omega(n \log n)$	[4]
*	Approximate	Standard	*	*	$\Omega(n \log n)$	[4]
Randomized	Exact		Approximate	Non-contiguous	$O(n)$	This paper
Randomized	*	Weak	Approximate	Non-contiguous	$O(n)$	This paper

Table 1: Summary of known results. An asterisk means that the result holds for both choices.

In the analogous multiple-choice balls and bins model, each player independently selects d' of αn discrete bins uniformly at random. This balls and bins model is equivalent to the special case of the cake model in which all the partitions are identical. It is a folklore result that in the balls and bins model, the maximum load is $\Theta(\log n)$ if $d' = 1$; And if $d' > 1$, then with one can with high probability pick one of the d' pieces for each player in such a way that each bin only has 1 ball. One can even get maximum load $O(\log \log n)$ if the assignment has to be made online player by player [2].

Analysis of balls and bins models have found wide application in areas such as load balancing [8]. In these situations, a ball represents a job that can be assigned to various bins/machines. Roughly speaking, load balancing of identical machines is to balls and bins, as load balancing on unrelated machines is to cake cutting. Unrelated machines is one of the standard models in the load balancing literature [1]. In the unrelated machines model there is a speed $s_{i,j}$ that a machine i can work on a job j. Assume that jobs can use more than one machine, and that machines can be shared. Then the total value of the machines to job j is $\sum_i s_{i,j}$, and a c-fair allocation for job j would be a collection of machines, or portions of machines, that can together process j at a speed of $\sum_i \frac{s_{i,j}}{cn}$. So it seems to us reasonable to presume the the cake model, and balanced allocation lemmas, should have interesting applications in settings involving load balancing on unrelated machines.

We now briefly discuss how our Balanced Allocation Lemma can be used to solve the cake cutting problem (See Appendix Section 4 for more details). The i^{th} candidate piece is the i^{th} subinterval of value $\frac{1}{\alpha n}$, which can be found by two cut queries. After the application of the Balanced Allocation Lemma, any standard fair allocation algorithm can be used to divide any portion of the cake desired by more than one player.

1.1 Related Results

The first step towards obtaining an $\Omega(n \log n)$ lower bound on the complexity of cake cutting was taken by Magdon-Ismail, Busch, and Krishnamoothy [7], who were able to show that any protocol must make $\Omega(n \log n)$ comparisons to compute the assignment. So this result does not address query complexity. Approximately fair protocols were introduced by Robertson and Webb [9]. Traditionally, much of the research has focused on minimizing the number of cuts, presumably out of concern that too many cuts would lead to crumbling of a literal cake. There is a deterministic protocol that achieves $O(1)$-fairness with $\Theta(n)$ cuts and $\Theta(n^2)$ evaluations [9, 6, 13]. There are several other objectives studied in the cake cutting setting, most notably, max-min fairness, and envy-free fairness.

The literature on balanced allocations is also rather large. A nice survey is given in [8]. We are not aware of any other results on balanced allocations for unrelated machines.

2 Intuition

In this section we try to give some intuition and a road map for the proof of our Balanced Allocation Lemma. We start with an example instance, see Figure 1 that demonstrates several interesting features of the cake model and our analysis. Each of the rows consists of the αn subintervals of the n players. The $n/2$ A players have αn candidate pieces of identical length. Then for $i \in [1, \sqrt{\frac{n}{2}}]$, there is a group of $\sqrt{\frac{n}{2}}$ B_i players. Half of a B_i's candidate pieces overlap with the $2i^{th}$ piece of the A players, and half with the $2i+1^{st}$ piece of the A players.

Figure 1: An example in which player's intervals overlap in more complex ways.

One immediate observation is that maximum load equal to 1 result from the standard multiple-choice balls and bins model will not carry over to the cake model. To see this, note that with high probability, one of the A players chooses all of his d' pieces from his first $2\sqrt{\frac{n}{2}}$ candidate pieces. Call this player A'. Also with high probability, for each d' pieces of A', there is a B_i player that has all of d' pieces overlapping with it. This explains the need to relax the maximum load bound from 1 to $O(1)$.

The Implication Graph: To gain intuition, let us assume for the moment that $d' = 2$. Let $c_{\langle p,i \rangle}$ denote the $i^{th} \in [1, \alpha n]$ candidate piece for player p. Let $a_{\langle p,0 \rangle}$ and $a_{\langle p,1 \rangle}$ be the two semifinal pieces selected for player p. We now define what we call the implication graph. The vertices of the implication graph are the $2n$ pieces $a_{\langle p,r \rangle}$, $1 \leq p \leq n$ and $0 \leq r \leq 1$. If piece $a_{\langle p,r \rangle}$ intersects piece $a_{\langle q,s \rangle}$, then there is an directed edge from piece $a_{\langle p,r \rangle}$ to piece $a_{\langle q,1-s \rangle}$ and similarly from $a_{\langle q,s \rangle}$ to $a_{\langle p,1-r \rangle}$. The intuition is that if player p gets $a_{\langle p,r \rangle}$ as his final piece, then player q must get piece $a_{\langle q,1-s \rangle}$ if p and q's pieces are not going to overlap. Similarly if q gets $a_{\langle q,s \rangle}$, then p must get $a_{\langle p,1-r \rangle}$. As an example, Figure 2 gives a subset of the semifinal pieces selected from the candidate pieces in Figure 1. The directed edges arising from this example are given.

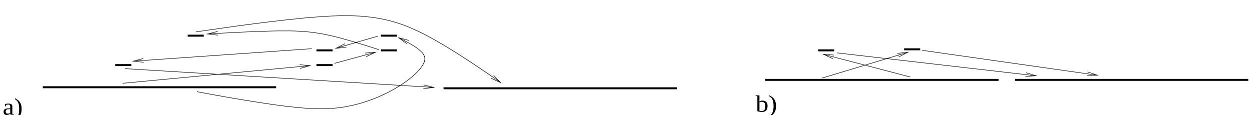

Figure 2: Two excerpts from an implication graph.

Pair Path: We define a *pair path* in the implication graph to be a directed path between the two pieces for one player, i.e. from some $a_{\langle p,r \rangle}$ to $a_{\langle p,1-r \rangle}$. In Figure 2.a, there are two such paths of length four from the A player's left semifinal piece to his right and in Figure 2.b two paths of length two. Note that such paths are problematic because they effectively say that if the A player gets his left semifinal piece as his final piece then he must get his right piece, which of course implies that he must get his right piece. We will show that if the implication graph G does not contain any such pair paths, then the following algorithm selects a final piece for each player in such a way that these final pieces are disjoint. (See Section 4.1.)

Final Piece Selection Algorithm Description: We repeatedly pick an arbitrary player p that has not selected a final piece. We pick the piece $a_{\langle p,0\rangle}$ as the final piece for p. Further, we pick as final pieces all those pieces in G that are reachable from $a_{\langle p,0\rangle}$ in G.

Independent Edges: To gain intuition, we now sketch a proof that the implication graph does not contain a pair path for the balls and bins model (each player's collection of αn candidate pieces are identical). Note that in the balls and bins model, every pair path has to be of length at least 3. Consider a possible pair path $a_{\langle p_0,r_0\rangle}, a_{\langle p_1,r_1\rangle}, \ldots a_{\langle p_{k-1},r_{k-1}\rangle}, a_{\langle p_0,1-r_0\rangle}$ with k edges in the implication graph. The probability that a particular pair of nodes $\left\langle a_{\langle p_0,r_0\rangle}, a_{\langle p_1,r_1\rangle}\right\rangle$ has an edge between them, i.e. the probability that the candidate piece chosen to be $a_{\langle p_0,r_0\rangle}$ intersects with that chosen to be $a_{\langle p_1,1-r_1\rangle}$, is $\frac{1}{\alpha n}$. The presence or absence of these k edges in the implication graph are statistically independent. Thus the probability that this particular pair path appears in the implication graph is at most $\left(\frac{1}{\alpha n}\right)^k$. Since there are at most $\binom{2n}{k}k!$ possible pair paths with k edges, the probability that there is pair path is at most $\sum_{k=3}^{n}\binom{2n}{k}k!\frac{1}{(\alpha n)^k}$. If α is sufficiently large, then this probability is say at most $1/2$.

We now return to the general cake model. One difficulty is that the edges in the implication graph are no longer independent. To see this, recall Figure 1. The probability that any two semifinal pieces over lap is still $O(\frac{1}{\alpha n})$. However, if one of an A player's semifinal pieces overlaps with one B_i player's semifinal piece, then we know that this A player must have selected either his $2i^{th}$ or $2i+1^{st}$ candidate piece and hence it very likely to also overlap with another B_i player's semifinal piece.

Pair Paths of Length $\geq$ Three and Vees: Such dependencies can occur when there is what we call a vee among the candidate pieces. We define a *vee* to consist of a triple of pieces, one *center* piece and two *base* pieces, with the property that the center piece intersects both of the base two pieces. For example, see the three left most pieces in Figure 2.a.

Note that in the balls and bins model, the expected number of vee's among the semifinal pieces is $O(\binom{2n}{3}\frac{1}{(\alpha n)^2}) = O(n)$. And in the cake model, we will show that if the expected number of vee's among the semifinal pieces is $O(n)$, then with probability $\Omega(1)$ there will be no pair path with three of more edges in the implication graph of the semifinal pieces. (See Section 4.3). Unfortunately, in the example in Figure 1, it is the case that, with high probability, the number of vees among the semifinal pieces is $\Omega(\sqrt{n}\cdot(\sqrt{n})^2) = \Omega(n^{3/2})$. The consequence of this is that, with high probability, there will be pair paths like those in Figure 2.a. One can also construct instances where the number of vees is $\Omega(n^2)$ with probability $\Omega(1)$.

Getting the expected number of vee's in the semifinal pieces down to $O(n)$ necessitates that $d' \geq 4$. Let us now explain how we accomplish this. The selection of final pieces will occur in three instead of two phases. First, each player independently at randomly chooses $d' = 2d$ *quarterfinal* pieces. These are partitioned into two *brackets* $A_{\langle p,0\rangle}$ and $A_{\langle p,1\rangle}$ containing d pieces each. From each such bracket, we choose one interval, denoted $a_{\langle p,r\rangle}$ to be a *semifinal* piece. The semifinal piece is chosen to be the one that intersects the smallest number of other candidate pieces, $c_{\langle q,j\rangle}$. Note that this processes is independent for the different players p and for each bracket. We will show then that the expected number of vees in the resulting $2n$ semifinal pieces is $O(n)$ (see Section 4.2). We show that as a consequence of this, with probability $\Omega(1)$, the implication graph of the semifinal pieces does not contain a pair path of length 3 or longer.

Pair Paths of Length Two and Same-Player-Vees: Another difficulty is that the implication graph of the semifinal pieces may, with high probability, have pair paths of length two. See Figure 2.b. A pair path of length two occurs if and only if the implication graph contains what we call a same-player-vee. A *same-player-vee* is a vee where both of the base pieces belong to the same player. That is, there is a center piece $a_{\langle p,r\rangle}$ and two bases $a_{\langle q,0\rangle}$ and $a_{\langle q,1\rangle}$. In the instance in Figure 1, it is the case that with high probability there will be many same-player-vees.

To get around the problem of same-player-vees, we introduce the *same-player-vee graph* with directed edge $\langle p, q\rangle$ when these players are involved in a same-player-vee. We show that with probability $\Omega(1)$ there are no paths in this graph containing $w = 2$ edges. Hence the same-player-vee graph can be colored with 2 colors. (See Section 4.4). Therefore, with probability $\Omega(1)$, we can partition the players into 2 partitions in such a way there is no same-player-vee involving two players in the same partition.

Summary of Balanced Allocation Algorithm: We summarize our Balanced Allocation Algorithm.

- Independently, for each player $p \in [1, n]$ and each $r \in [0, 1]$, randomly choose d of the candidate pieces $c_{\langle p,i\rangle}$ to be in the quarterfinal bracket $A_{\langle p,r\rangle}$.
- In each quarterfinal bracket $A_{\langle p,r\rangle}$, pick as the semifinal piece $a_{\langle p,r\rangle}$, the piece that intersects the fewest other candidate pieces $c_{\langle q,j\rangle}$. If we are unlucky and the Implication Graph contains a pair path of length greater than 3, then start over. See Sections 4.2 and 4.3.
- Construct and vertex color the same-player-vee graph using the greedy coloring algorithm using at most $w = 2$ colors. See Section 4.4. Let S_k be the subgraph of the implication graph containing only those players colored k. This ensures that Implication Graph restricted to S_k contains no pair paths of length 2.
- For each S_k, pick the final piece for each player involved in S_k by applying the Final Piece Selection Algorithm to S_k. See Section 4.1. Because the Implication Graph on S_k contains no pair paths of any length, this algorithm ensures that these final pieces for each player are disjoint, i.e. for any point in the cake, the final piece of at most one player from S_k covers this point.
- Conclude that for any point in the cake, the final piece of at most $w = 2$ players cover this point. The total probability of success is computed in Section 4.5.

In section 4.6 we extend this Balanced Allocation Algorithm to the case of approximate queries against a weak adversary.

3 The Proofs

In this section we prove the various claims that we made in the previous section. Each subsection can essentially be read independently of the others.

4 Our Cake Cutting Algorithm

Before turning to our Balanced Allocation Lemma, let us explain how our cake cutting protocol uses our Balanced Allocation Algorithm. Each player p has an initially unknown value function V_p that specifies how much that player values each subinterval of the unit interval. We imagine the player partitioning the cake into αn pieces each of value $\frac{1}{\alpha n}$. The i^{th} such candidate piece of cake $c_{\langle p,i\rangle}$ can be obtained using the two queries $(Cut_p(0, \frac{i-1}{\alpha n}), Cut_p(0, \frac{i}{\alpha n}))$. Our cake cutting protocol uses our Balanced Allocation Algorithm to obtain a final piece for each player such that every point of the cake is covered by at most $O(1)$ of these final pieces. Because each player chooses only a constant number of candidate pieces, the query complexity is $\Theta(n)$. Because the probability of success is $\Theta(1)$, they expect to repeat it $\Theta(1)$ times until they succeed. Once each player has one final piece, we need to divide these pieces further so that the players have disjoint collections of cake intervals. This is done as follows. These n final pieces have $2n$ endpoints and these endpoints partition the cake into $2n$ pieces. Denote these by f_j. For each piece f_j and each player p, the player either wants all of f_j or none of it. For each j, let S_j be the set of players wanting cake piece f_j.

Some players p may appear in more than one S_j, but we have that $|S_j| \leq k = O(1)$, because every point of the cake is covered by at most $O(1)$ of player's final pieces. For each piece f_j, the players in S_j use any fair algorithm to partition f_j between them. Each such application has complexity $\Theta(1)$ since it only involves $\Theta(1)$ players. This protocol guarantees $k\alpha$-fairness. Consider player p. For each j for which $p \in S_j$, let $v_{\langle p,j \rangle}$ denote the amount he values piece f_j. Note $\sum_j v_{\langle p,j \rangle} = V_p(\cup_j f_j) = V_p(\text{his final piece}) = \frac{1}{\alpha n}$. When fairly dividing f_j, he receives a piece of f_j with value at least $\frac{v_{\langle p,j \rangle}}{k}$. The total cake that he receives has total value $\sum_j \frac{v_{\langle p,j \rangle}}{k} = \frac{1}{k\alpha n}$. Note that unlike all previous cake cutting algorithms, this one does not guarantee contiguous portions since a player's final interval may be involved many different such subintervals f_j.

4.1 Final Piece Selection Algorithm

We show some structural properties of the implication graph imply the correctness of the Final Piece Selection Algorithm.

In this subsection, we show how to select from each player's two semifinal pieces $a_{\langle p,0 \rangle}$ and $a_{\langle p,1 \rangle}$, one final piece for each player. Recall that if semifinal pieces $a_{\langle p,r \rangle}$ piece $a_{\langle q,s \rangle}$ intersect, then there is an directed edge in the implication graph G from $a_{\langle p,r \rangle}$ to $a_{\langle q,1-s \rangle}$ and similarly from $a_{\langle q,s \rangle}$ to $a_{\langle p,1-r \rangle}$. The intuition is that if player p gets $a_{\langle p,r \rangle}$ as his final piece, then player q must get piece $a_{\langle q,1-s \rangle}$ if p and q's pieces are not going to overlap. Also recall that a pair path in this graph is a directed path between the pair of pieces for one player, i.e. from some $a_{\langle p,r \rangle}$ to $a_{\langle p,1-r \rangle}$. In the final piece selection algorithm we repeatedly pick an arbitrary player p that has not selected a final piece and give him the piece $a_{\langle p,0 \rangle}$. Further, we pick as final pieces all those pieces in G that are reachable from $a_{\langle p,0 \rangle}$ in G. The goal of this subsection is the following lemma.

Lemma 2. *If there is a path in G from $a_{\langle p,r \rangle}$ to $a_{\langle q,s \rangle}$ then there must be a path from $a_{\langle q,1-s \rangle}$ to $a_{\langle p,1-r \rangle}$ in G.*

Proof. By the way two edge are added at once, if there is an edge from $a_{\langle p,r \rangle}$ to $a_{\langle q,s \rangle}$ then there must be an edge from $a_{\langle q,1-s \rangle}$ to $a_{\langle p,1-r \rangle}$ in G. The lemma then follows by induction. □

Lemma 3. *If both the pieces $a_{\langle q,0 \rangle}$ and $a_{\langle q,1 \rangle}$ are reachable from a piece $a_{\langle p,r \rangle}$ in the implication graph G, then G has a pair path.*

Proof. By Lemma 2, the existence of the path from $a_{\langle p,r \rangle}$ to $a_{\langle q,0 \rangle}$ implies the existence of a path from $a_{\langle q,1 \rangle}$ to $a_{\langle p,1-r \rangle}$. But this means that there is a path from $a_{\langle p,r \rangle}$ to $a_{\langle q,1 \rangle}$ to $a_{\langle p,1-r \rangle}$. □

Lemma 4. *If an implication graph G of the semifinal pieces does not contain a pair path, then the Final Piece Selection Algorithm selects a final piece for each player and these final pieces are disjoint.*

Proof. Consider an iteration that starts by assigning $a_{\langle p,0 \rangle}$ to player p. This iteration will force the assignment of at most one piece to any one player because by Lemma 3 there can not be a player q such that both $a_{\langle q,0 \rangle}$ and $a_{\langle q,1 \rangle}$ are reachable from $a_{\langle p,0 \rangle}$. Similarly, if this same iteration forces player q to be assigned say to $a_{\langle q,0 \rangle}$, then we need to prove that he has not already been assigned $a_{\langle q,1 \rangle}$ during an earlier iteration. If assigning $a_{\langle p,0 \rangle}$ forces $a_{\langle q,0 \rangle}$, then there is a path from the one to the other. Hence, by Lemma 2, there is a path from $a_{\langle q,1 \rangle}$ to $a_{\langle p,1 \rangle}$. Hence, if $a_{\langle q,1 \rangle}$ had been previously assigned, then player p would have been forced to $a_{\langle p,1 \rangle}$ and in this case p would not be involved in this current iteration. The disjointness of the final pieces follows from the definition of the implication graph. □

4.2 The Number of Vees

In this subsection we show that the number of vees is $O(n)$ with probability $\Omega(1)$. Recall that a *vee* consists of a triple of semifinal pieces, one *center* piece $a_{\langle p,r\rangle}$ and two *base* pieces $a_{\langle q,s\rangle}$ and $a_{\langle q',s'\rangle}$, with the property that the center piece intersects both of the base two pieces.

Lemma 5. *Assume that m players have partitioned their cake into αn pieces each. Let $\ell_{p,i}$ be the number of pieces of the other players that overlap with piece i of player p. Then for any player p, $\sum_{i=1}^{\alpha n} \ell_{p,i} \leq 2\alpha nm$.*

Lemma 6. *The probability that semifinal piece $a_{\langle p,r\rangle}$ overlaps with semifinal piece $a_{\langle q,s\rangle}$ is at most $\frac{2d^2}{\alpha n}$.*

Proof. The number of pairs $\{i,j\}$ such that candidate piece $c_{\langle p,i\rangle}$ intersects candidate piece $c_{\langle q,j\rangle}$ is at most $2\alpha n$. When d of the αn candidate pieces for player p are selected to be the quarterfinal pieces in $A_{\langle p,r\rangle}$, the probability that $c_{\langle p,i\rangle}$ is among them is $\frac{d}{\alpha n}$. Hence, the probability that the semifinal piece $a_{\langle p,r\rangle}$ is selected to be this candidate piece $c_{\langle p,i\rangle}$ is at most this. Similarly, that $a_{\langle q,s\rangle}$ is $c_{\langle p,i\rangle}$. Finally, the probability that $a_{\langle p,r\rangle}$ overlaps with $a_{\langle q,s\rangle}$ is than at most $2\alpha n \cdot \frac{d}{\alpha n} \cdot \frac{d}{\alpha n} = \frac{2d^2}{\alpha n}$. □

Lemma 7. *The expected number of vee's in G is at most $\frac{16d^3}{\alpha^2}n$.*

Proof. Consider a particular player p. Again let $\ell_{\langle p,i\rangle}$ denote the total number of candidate pieces overlapping the i^{th} candidate piece $c_{\langle p,i\rangle}$ of the player p. Without loss of generality, let us renumber p's candidate pieces in non-increasing order by $\ell_{\langle p,i\rangle}$, that is, $\ell_{\langle p,i\rangle} \geq \ell_{\langle p,i+1\rangle}$.

For $p \in [n]$, $i \in [\alpha n]$, and $r \in [0,1]$, let $R_{\langle p,i,r\rangle}$ be the event that the candidate $c_{\langle p,i\rangle}$ is selected to be the semifinal piece $a_{\langle p,r\rangle}$. To understand this, let us review how this is chosen. First, player p randomly chooses d candidate pieces to be in his quarterfinal brackets $A_{\langle p,r\rangle}$. Then the semifinal piece $a_{\langle p,r\rangle}$ is chosen to be the one with the smallest $\ell_{\langle p,i\rangle}$ value or, by our ordering, the one with the largest index. Hence, the probability of $R_{\langle p,i,r\rangle}$ is the probability that d indexes are randomly selected from an indexes and the largest selected index is i. This gives $\text{Prob}[R_{\langle p,i,r\rangle}] = d \cdot (\frac{1}{\alpha n}) \cdot (\frac{i-1}{\alpha n})^{d-1}$.

Let $x_{\langle p,r\rangle}$ be the number of vee's with $a_{\langle p,r\rangle}$ as the center. There are $\binom{\ell_{\langle p,i\rangle}}{2}$ pairs of candidate pieces that might be the two base pieces $a_{\langle q,s\rangle}$ and $a_{\langle q',s'\rangle}$ with the center piece $a_{\langle p,r\rangle} = c_{\langle p,i\rangle}$. The probability that both of this pair are semifinal pieces is at most $\left(\frac{2d}{\alpha n}\right)^2$. Hence, $E[x_{\langle p,r\rangle} \mid R_{\langle p,i,r\rangle}]$ is at most $\binom{\ell_{\langle p,i\rangle}}{2}\left(\frac{2d}{\alpha n}\right)^2 \leq 2\ell_{\langle p,i\rangle}^2 \left(\frac{d}{\alpha n}\right)^2$.

$$
\begin{aligned}
E[x_{\langle p,r\rangle}] &= \sum_{i=1}^{\alpha n} \text{Prob}[R_{\langle p,i,r\rangle}] \cdot E[x_{\langle p,r\rangle} \mid R_{\langle p,i,r\rangle}] \leq \sum_{i=1}^{\alpha n} \left(\frac{d}{\alpha n}\right)\left(\frac{i-1}{\alpha n}\right)^{d-1} \cdot 2\ell_{\langle p,i\rangle}^2 \left(\frac{d}{\alpha n}\right)^2 \\
&\leq \left(\frac{2d^3}{(\alpha n)^{d+2}}\right) \cdot \sum_{i=1}^{\alpha n} i^{d-1} \ell_{\langle p,i\rangle}^2
\end{aligned}
$$

Lemma 5 bounds that $\sum_{i=1}^{\alpha n} \ell_{\langle p,i\rangle} \leq 2\alpha n^2 = M$. The next lemma then bounds $\sum_{i=1}^{m} i^{d-1}\ell_{\langle p,i\rangle}^2 \leq m^{d-2}M^2$.

$$
E[x_{\langle p,r\rangle}] \leq \left(\frac{2d^3}{(\alpha n)^{d+2}}\right) \cdot (\alpha n)^{d-2} \cdot \left(2\alpha n^2\right)^2 \leq \frac{8d^3}{\alpha^2}.
$$

By linearity of expectation, the expected number of vees over all is $\sum_{p=1}^{n}\sum_{r=0}^{1} E[x_{\langle p,r\rangle}] \leq 2n \cdot \frac{8d^3}{\alpha^2}$. □

Lemma 8. *If $d \geq 2$, $\forall i \in [1, m-1]$ $\ell_i \geq \ell_{i+1} \geq 0$, and $\sum_{i=1}^m \ell_i = M$, then $\sum_{i=1}^m i^{d-1}\ell_i^2 \leq m^{d-2}M^2$.*

Proof. We will start with some intuition. To maximize, $\sum_{i=1}^m i^{d-1}\ell_i^2$, one wants to make the ℓ in the last term as big possible. By example, if $\ell_m = M$, then $\sum_{i=1}^m i^{d-1}\ell_i^2$ would be $m^{d-1}M^2$, which is a factor of m more than we need to get the number of vees down from $\Theta(n^2)$ to $O(n)$. However, the constraint $\ell_i \geq \ell_{i+1}$ prevents this example and pressures all the ℓ's to be the same. If they are the same, then $\ell_i = \frac{M}{m}$ and $\sum_{i=1}^m i^{d-1}\ell_i^2 = O(m^d(\frac{M}{m})^2)$ as required. We need some definitions to prove this formally. Let $\ell_{m+1} = 0$, and $s_i = \ell_i - \ell_{i+1}$ for $1 \leq i \leq m$. Note that our constraint gives that $s_i \geq 0$. Further more, $\ell_i = \sum_{j=i}^m s_j$ and $M = \sum_{i=1}^m \ell_i = \sum_{i=1}^m is_i$. Then let $t_i = is_i$ so that $M = \sum_{i=1}^m t_i$. Now using basic algebra we conclude that

$$\begin{aligned}
\sum_{i=1}^m i^{d-1}\ell_i^2 &= \sum_{i=1}^m i^{d-1}\left(\sum_{j=i}^m s_j\right)^2 = \sum_{i=1}^m i^{d-1}\sum_{j=i}^m\sum_{k=i}^m s_j s_k = \sum_{j=1}^m\sum_{k=1}^m s_j s_k \sum_{i=1}^{\min(j,k)} i^{d-1} \\
&\leq \sum_{j=1}^m\sum_{k=1}^m \frac{t_j t_k}{jk}\min(j,k)^d \leq m^{d-2}\sum_{j=1}^m\sum_{k=1}^m t_j t_k = m^{d-2}\left(\sum_{j=1}^m t_j\right)^2 = m^{d-2}M^2
\end{aligned}$$

□

4.3 The Existence of Pair Paths

In this subsection, we show that with probability $\Omega(1)$, the implication graph doesn't contain a pair path of length three of more. Recall that if the semifinal pieces $a_{\langle p,r\rangle}$ and $a_{\langle q,s\rangle}$ intersect, then there is an directed edge in the implication graph G from $a_{\langle p,r\rangle}$ to $a_{\langle q,1-s\rangle}$ and from $a_{\langle q,s\rangle}$ to $a_{\langle p,1-r\rangle}$ and that a *pair path* is a directed path between the two semifinal pieces for the same player, i.e. from some $a_{\langle p,r\rangle}$ to $a_{\langle p,1-r\rangle}$. The next lemma is best understood by studying Figure 3.

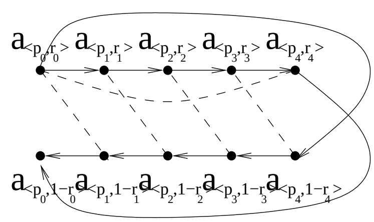

Figure 3: A pair path within the implication graph is given. A pair of nodes in the same column represents the pairs of semi-final pieces for one player. The dotted edges are between semi-final pieces that overlap. The solid directed edges are the resulting edges in the implication graph.

Lemma 9. *Consider a simple pair path $P = \langle a_{\langle p_0,r_0\rangle}, a_{\langle p_1,r_1\rangle}, \ldots a_{\langle p_{k-1},r_{k-1}\rangle}, a_{\langle p_0,1-r_0\rangle}\rangle$ of length $k \geq 3$. Let V be the vee with center $a_{\langle p_0,r_0\rangle}$ and bases $a_{\langle p_1,1-r_1\rangle}$ and $a_{\langle p_{k-1},r_{k-1}\rangle}$. For $i \in [1, k-2]$, let $I_i \in G$ be the event that semifinal pieces $a_{\langle p_i,r_i\rangle}$ and $a_{\langle p_{i+1},1-r_{i+1}\rangle}$ intersect. Then*

$$\text{Prob}[P \in G] \leq \text{Prob}[V \in G] \cdot \Pi_{i=1}^{k-2}\text{Prob}[I_i \in G]$$

Proof. The edges from $a_{\langle p_0,r_0\rangle}$ to $a_{\langle p_1,r_1\rangle}$ and from $a_{\langle p_{k-1},r_{k-1}\rangle}$ to $a_{\langle p_0,1-r_0\rangle}$ mean that $a_{\langle p_0,r_0\rangle}$ intersect with both $a_{\langle p_1,1-r_1\rangle}$ and $a_{\langle p_{k-1},r_{k-1}\rangle}$. Hence, the vee V occurs. The edge from $a_{\langle p_i,r_i\rangle}$ to $a_{\langle p_{i+1},r_{i+1}\rangle}$ means that

$a_{\langle p_i, r_i\rangle}$ and $a_{\langle p_{i+1}, 1-r_{i+1}\rangle}$ intersect, i.e. I_i. It follows that $\mathrm{Prob}[P \in G] \le \mathrm{Prob}[V \;\&\; \text{each } I_i \in G]$. What remains is to prove that the events V and each I_i are independent. Whether a semifinal piece of players p and q intersect is independent of whether a semifinal piece of different players p' and q' intersect because these event have nothing to do with each other. This remains true when the players p and p' are the same, but the we are talking about different semifinal pieces of this player, namely event I_i and I_{i+1} are independent. This is because the selection of the quarterfinal pieces for the bracket $A_{\langle p,0\rangle}$ and the selection of p's semifinal piece $a_{\langle p,0\rangle}$ within this bracket is independent of this process for his other semifinal piece $a_{\langle p,1\rangle}$. □

Lemma 10. *The probability that the implication graph G contains a pair path of length at least three is at most $\frac{32d^5}{\alpha^2(\alpha-4d^2)}$.*

Proof. Let $\mathcal{V}$ be the set of all 3-tuples representing all possible vee's in G and for $V \in \mathcal{V}$ let $\mathcal{P}_k(V)$ be the set of all possible pair paths of length k that include the vee V. The probability that G contains a pair path of length at least three is at most

$$\sum_{k=3}^{n} \sum_{V\in\mathcal{V}} \sum_{P\in\mathcal{P}_k(V)} \mathrm{Prob}[P \in G] \tag{1}$$

$$\le \sum_{k=3}^{n} \sum_{V\in\mathcal{V}} \sum_{P\in\mathcal{P}_k(V)} \mathrm{Prob}[V \in G] \cdot \Pi_{i=1}^{k-2}\mathrm{Prob}[I_i \in G] \tag{2}$$

$$\le \sum_{k=3}^{n} \sum_{V\in\mathcal{V}} \mathrm{Prob}[V \in G] \sum_{P\in\mathcal{P}_k(V)} \left(\frac{2d^2}{\alpha n}\right)^{k-2} \tag{3}$$

$$\tag{4}$$

$$\le \sum_{k=3}^{n} \sum_{V\in\mathcal{V}} \mathrm{Prob}[V \in G] \left(\binom{2n}{k-3}(k-3)!\right) \left(\frac{2d^2}{\alpha n}\right)^{k-2} \tag{5}$$

$$\le \sum_{k=3}^{n} (2n)^{k-3} \left(\frac{2d^2}{\alpha n}\right)^{k-2} \sum_{V\in\mathcal{V}} \mathrm{Prob}[V \in G] \tag{6}$$

$$\le \sum_{k=3}^{n} (2n)^{k-3} \left(\frac{2d^2}{\alpha n}\right)^{k-2} \left(\frac{16d^3}{\alpha^2} n\right) \tag{7}$$

$$\le \frac{8d^3}{\alpha^2} \sum_{k=3}^{n} \left(\frac{4d^2}{\alpha}\right)^{k-2} \le \frac{8d^3}{\alpha^2} \left(\frac{4d^2}{\alpha}\right) \left(\frac{1}{1-4d^2/\alpha}\right) = \frac{32d^5}{\alpha^2(\alpha-4d^2)} \tag{8}$$

The inequality in line 2 follows from Lemma 9 and line 3 from Lemma 6. The inequality in line 5 holds since there are $k-3$ pieces in P that are not part of the vee V. The inequality in line 7 follows from Lemma 7. □

4.4 Coloring Same-Player-Vee Graphs

In this subsection we show that with probability $\Omega(1)$, we can color the same-player-vee graph with 2 colors since this graph will have no paths of length $w = 2$.

Lemma 11. *The probability that the same-player-vee graph is not $w = 2$ colorable is at most $\frac{16d^3}{\alpha^3} + \frac{8d^2}{\alpha^2}$.*

Recall that we put the directed edge $\langle p, q\rangle$ in the same-player-vee graph if one of player p's two semifinal pieces, namely $a_{\langle p,0\rangle}$ or $a_{\langle p,1\rangle}$, overlap with both of player q's two semifinal pieces, namely $a_{\langle q,0\rangle}$ and $a_{\langle p,1\rangle}$. Hence, a path of length 3 consists of semi-final pieces $a_{\langle p_1,r_1\rangle}$, $a_{\langle p_2,r_2\rangle}$, $a_{\langle p_2,1-r_2\rangle}$, $a_{\langle p_3,0\rangle}$, and $a_{\langle p_3,1\rangle}$ for three players p_1, p_2, and p_3, where both $a_{\langle p_2,r_2\rangle}$ and $a_{\langle p_2,1-r_2\rangle}$ overlap with $a_{\langle p_1,r_1\rangle}$, and both $a_{\langle p_3,0\rangle}$ and $a_{\langle p_3,1\rangle}$ overlap with $a_{\langle p_2,r_2\rangle}$. We will consider the probability of such paths starting backwards.

Lemma 12. *Suppose we are considering a set of $\widehat{\ell}$ candidate pieces for the semi-final pieces $a_{\langle p_3,0\rangle}$ and $a_{\langle p_3,1\rangle}$. The probability that some player gets both of his semi final pieces from this set is at most* $\min((\frac{d\widehat{\ell}}{\alpha n})^2, 1)$.

Proof. If $\widehat{\ell} \leq \alpha n$, then all $\widehat{\ell}$ pieces in this set can be the candidate pieces for one player p_3. The probability that both of his semi-final pieces are in this set is as stated. This probability, however, can never be bigger than one. ☐

Consider some candidate piece $c_{\langle p_1,i\rangle}$ that potentially might be $a_{\langle p_1,r_1\rangle}$. Let $\ell_{\langle p_1,i\rangle}$ denote the number of other candidate pieces of overlapping it. Consider some player p_2. Let $c_{\langle p_2,j_l\rangle}, c_{\langle p_2,j_l+1\rangle}, \ldots, c_{\langle p_2,j_r\rangle}$ be the candidate pieces of player p_2 that overlap with piece $c_{\langle p_1,i\rangle}$. Let $\ell_{\langle p_2,j\rangle}$ denote the number of other candidate pieces of overlapping $c_{\langle p_2,j\rangle}$. What we would like to be true is $\sum_{j=j_l}^{j_r} \ell_{\langle p_2,j\rangle} = \ell_{\langle p_1,i\rangle}$. This is true in Figure 4.A.

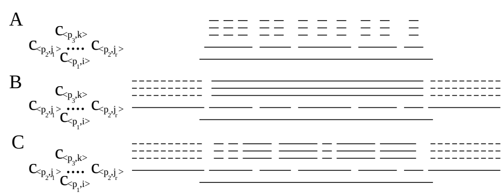

Figure 4: An pair path within the implication graph is given. Pair of nodes in columns represent the pairs of semi-final pieces for a player. The dotted edges are between semi-final pieces that overlap. The solid directed edges are the resulting edges in the implication graph.

However, as seen in Figure 4.B, the left and right most pieces $c_{\langle p_2,j_l\rangle}$ and $c_{\langle p_2,j_r\rangle}$ might have $\ell_{\langle p_2,j_l\rangle}$ and $\ell_{\langle p_2,j_l\rangle}$ as big as αn^2 and even excluding these $\sum_{j=j_l+1}^{j_r-1} \ell_{\langle p_2,j\rangle}$ might be as big as $(j_r - j_l - 1) \cdot \ell_{\langle p_1,i\rangle}$. Towards proving something like $\sum_{j=j_l}^{j_r} \ell_{\langle p_2,j\rangle} = \ell_{\langle p_1,i\rangle}$, Consider some player p_3. Define $\ell_{\langle p_2,j,p_3\rangle}$ to be the number of player p_3's candidate pieces that overlap $c_{\langle p_2,j\rangle}$. Note that if $\ell_{\langle p_2,j,p_3\rangle} = 1$, then it is impossible to have both of player p_3's semi-final pieces over lap with for $c_{\langle p_2,j\rangle}$. Hence, we can ignore player p_3 when considering $c_{\langle p_2,j\rangle}$ as being $a_{\langle p_2,r_2\rangle}$. Hence, define $\widehat{\ell}_{\langle p_2,j,p_3\rangle}$ to be $\ell_{\langle p_2,j,p_3\rangle}$ if $\ell_{\langle p_2,j,p_3\rangle} \geq 2$ and zero otherwise. Define $\widehat{\ell}_{\langle p_2,j\rangle} = \sum_q \widehat{\ell}_{\langle p_2,j,p_3\rangle}$. Note this is the number of pieces that overlap $c_{\langle p_2,j\rangle}$ excluding those pieces whose player only has one piece overlapping $c_{\langle p_2,j\rangle}$.

Lemma 13. *Then* $\sum_{i=j_l+1}^{j_r-1} \widehat{\ell}_{\langle p_2,j\rangle} \leq 2\ell_{\langle p_1,i\rangle}$.

Proof. First note that because we are not considering the left and right most candidate pieces $c_{\langle p_2,j_l\rangle}$ and $c_{\langle p_2,j_r\rangle}$ that overlap with $c_{\langle p_1,i\rangle}$, any piece that overlaps with $c_{\langle p_2,j\rangle}$ also overlaps with $c_{\langle p_1,i\rangle}$. (See Figure 4.C).

It is sufficient to prove that $\sum_{i=j_l+1}^{j_r-1} \widehat{\ell}_{\langle p_2,j,p_3\rangle}$ is at most twice the number of the $\ell_{\langle p_1,i\rangle}$ pieces that overlap $c_{\langle p_1,i\rangle}$ that belong to p_3. This is because for $\widehat{\ell}_{\langle p_2,j,p_3\rangle}$ to be non-zero, p_3's candidate pieces need to break at some point within piece $c_{\langle p_2,j\rangle}$. Hence, each of p_3's candidate pieces (that is counted) overlaps with at most two of the pieces $c_{\langle p_2,j\rangle}$ and $c_{\langle p_2,j+1\rangle}$. $\square$

Lemma 14. *Consider a candidate piece $c_{\langle p_1,i\rangle}$ such that there are $\ell_{\langle p_1,i\rangle}$ other candidate pieces overlapping it and some other player p_2. The probability that there are semi-final pieces $a_{\langle p_2,r_2\rangle}$, $a_{\langle p_3,0\rangle}$, and $a_{\langle p_3,1\rangle}$ for some player p_3, where $a_{\langle p_2,r_2\rangle}$ overlaps with $c_{\langle p_1,i\rangle}$, and both $a_{\langle p_3,0\rangle}$ and $a_{\langle p_3,1\rangle}$ overlap with $a_{\langle p_2,r_2\rangle}$ is at most $\frac{4d}{\alpha n} \cdot \left[\frac{d\ell_{\langle p_1,i\rangle}}{\alpha n} + 1\right]$.*

Proof. Consider a candidate piece $c_{\langle p_2,j\rangle}$ that overlaps with $c_{\langle p_1,i\rangle}$. The probability that candidate piece $c_{\langle p_2,j\rangle}$ is a semi-final piece for player p_2 is at most $\frac{2d}{\alpha n}$. By Lemma 12, the probability that there are semi-final pieces $a_{\langle p_3,0\rangle}$, and $a_{\langle p_3,1\rangle}$ for some player p_3 which both overlap with $c_{\langle p_2,j\rangle}$ is at most $\min((\frac{d\widehat{\ell}_{\langle p_2,j\rangle}}{\alpha n})^2, 1)$. It follows that the required probability is at most

$$\sum_{i=j_l}^{j_r} \frac{2d}{\alpha n} \cdot \min\left(\left(\frac{d\widehat{\ell}_{\langle p_2,j\rangle}}{\alpha n}\right)^2, 1\right) \leq \frac{2d}{\alpha n} \cdot \left[1 + \left[\sum_{i=j_l+1}^{j_r-1} \min\left(\left(\frac{d\widehat{\ell}_{\langle p_2,j\rangle}}{\alpha n}\right)^2, 1\right)\right] + 1\right].$$

By Lemma 13, $\sum_{i=j_l+1}^{j_r-1} \widehat{\ell}_{\langle p_2,j\rangle} \leq 2\ell_{\langle p_1,i\rangle}$. Hence, because of the quadratics in the sum, our sum is maximized by having a few $\widehat{\ell}_{\langle p_2,j\rangle}$ as big as possible. But because of the min, there is no reason to make a $\widehat{\ell}_{\langle p_2,j\rangle}$ bigger than $\frac{\alpha n}{d}$. Hence, the sum is maximized by setting $\frac{2d\ell_{\langle p_1,i\rangle}}{\alpha n}$ of the values $\widehat{\ell}_{\langle p_2,j\rangle}$ to $\frac{\alpha n}{d}$ and the rest to zero. This gives the result

$$\frac{2d}{\alpha n} \cdot \left[1 + \left[\frac{2d\ell_{\langle p_1,i\rangle}}{\alpha n} \cdot \min(1,1)\right] + 1\right].$$

$\square$

We will now add the requirement that player p_2's other candidate piece $a_{\langle p_2,1-r_2\rangle}$ also overlaps with $c_{\langle p_1,i\rangle}$ and sum the resulting probability over all possible players p_2.

Lemma 15. *Consider a candidate piece $c_{\langle p_1,i\rangle}$ such that there are $\ell_{\langle p_1,i\rangle}$ other candidate pieces overlapping it. The probability that there are semi-final pieces $a_{\langle p_2,r_2\rangle}$, $a_{\langle p_2,1-r_2\rangle}$, $a_{\langle p_3,0\rangle}$, and $a_{\langle p_3,1\rangle}$ for two players p_2 and p_3, where both $a_{\langle p_2,r_2\rangle}$ and $a_{\langle p_2,1-r_2\rangle}$ overlaps with $c_{\langle p_1,i\rangle}$, and both $a_{\langle p_3,0\rangle}$ and $a_{\langle p_3,1\rangle}$ overlap with $a_{\langle p_2,r_2\rangle}$ is at most $\frac{4d^2\ell^2_{\langle p_1,i\rangle}}{(\alpha n)^3} \cdot \left[1 + \frac{\alpha n}{d\ell_{\langle p_1,i\rangle}}\right]$.*

Proof. The probability that a particular candidate piece $c_{\langle p_2,j\rangle}$ is player p_2's semi-final piece $a_{\langle p_2,1-r_2\rangle}$ is at most $\frac{d}{an}$. Denote the number of player p_2's candidate pieces $c_{\langle p_2,j_l\rangle}, c_{\langle p_2,j_l+1\rangle}, \ldots, c_{\langle p_2,j_r\rangle}$ that overlap with piece $c_{\langle p_1,i\rangle}$ to be $q_{p_2} = j_r - j_l + 1$. Because these all overlap with $c_{\langle p_1,i\rangle}$, we have that $\sum_{p_2} q_{p_2} = \ell_{\langle p_1,i\rangle}$. Using Lemma 15, we get that the required probability is at most

$$\sum_{p_2} \frac{d}{an} \cdot q_{p_2} \cdot \left[\frac{4d}{\alpha n} \cdot \left[\frac{d\ell_{\langle p_1,i\rangle}}{\alpha n} + 1\right]\right] = \frac{d}{an} \cdot \ell_{\langle p_1,i\rangle} \cdot \left[\frac{4d}{\alpha n} \cdot \left[\frac{d\ell_{\langle p_1,i\rangle}}{\alpha n} + 1\right]\right] = \frac{4d^2\ell^2_{\langle p_1,i\rangle}}{(\alpha n)^3} \cdot \left[1 + \frac{\alpha n}{d\ell_{\langle p_1,i\rangle}}\right].$$

$\square$

We will now add the requirement that $c_{\langle p_1,i\rangle}$ is one of player p_1's semi-final pieces and sum up over all p_3 candidate pieces and over all players p_3.

Lemma 16. *The probability that there are semi-final pieces $a_{\langle p_1,r_1\rangle}$, $a_{\langle p_2,r_2\rangle}$, $a_{\langle p_2,1-r_2\rangle}$, $a_{\langle p_3,0\rangle}$, and $a_{\langle p_3,1\rangle}$ for three players p_1, p_2, and p_3, where both $a_{\langle p_2,r_2\rangle}$ and $a_{\langle p_2,1-r_2\rangle}$ overlap with $a_{\langle p_1,r_1\rangle}$, and both $a_{\langle p_3,0\rangle}$ and $a_{\langle p_3,1\rangle}$ overlap with $a_{\langle p_2,r_2\rangle}$ is at most $\frac{16d^3}{\alpha^3}+\frac{8d^2}{\alpha^2}$.*

Proof. As in the proof of Lemma 7, let $R_{\langle p,i,r\rangle}$ be the event that the candidate $c_{\langle p,i\rangle}$ is selected to be the semifinal piece $a_{\langle p,r\rangle}$. Recall that $\mathrm{Prob}[R_{\langle p,i,r\rangle}] = d\cdot(\frac{1}{\alpha n})\cdot(\frac{i-1}{\alpha n})^{d-1}$. There are n choices for player p_1. Thus by Lemma 15, our desired probability is at most

$$
\begin{aligned}
& n\left(\sum_{i=1}^{\alpha n}\frac{d}{\alpha n}\left(\frac{i-1}{\alpha n}\right)^{d-1}\frac{4d^2\ell^2_{\langle p_1,i\rangle}}{(\alpha n)^3}\cdot\left[1+\frac{\alpha n}{d\ell_{\langle p_1,i\rangle}}\right]\right)\\
\le\;& n\left(\frac{4d^3}{(\alpha n)^{d+3}}\sum_{i=1}^{\alpha n}\ell^2_{\langle p_1,i\rangle}(i-1)^{d-1}+\frac{4d^2}{(\alpha n)^{d+2}}\sum_{i=1}^{\alpha n}\ell_{\langle p_1,i\rangle}(i-1)^{d-1}\right)\\
\le\;& n\left(\frac{4d^3}{(\alpha n)^{d+3}}(\alpha n)^{d-2}(2\alpha n^2)^2+\frac{4d^2}{(\alpha n)^{d+2}}\sum_{i=1}^{\alpha n}\ell_{\langle p_1,i\rangle}(i-1)^{d-1}\right)\\
\le\;& n\left(\frac{4d^3}{(\alpha n)^{d+3}}(\alpha n)^{d-2}(2\alpha n^2)^2+\frac{4d^2}{(\alpha n)^{d+2}}(\alpha n)^{d}\left(\frac{2\alpha n^2}{\alpha n}\right)\right)=\frac{16d^3}{\alpha^3}+\frac{8d^2}{\alpha^2}
\end{aligned}
$$

The second inequality follows by Lemma 8. The third inequality follows from noting that, given that the $\ell_{\langle p_1,i\rangle}$'s are nonincreasing, the sum is obviously maximized if each $\ell_{\langle p_1,i\rangle}$ is equal. That is, each $\ell_{\langle p_1,i\rangle}=\frac{2\alpha n^2}{\alpha n}$. □

Proof. (Lemma 11) If a node of the same-player-vee graph is at the head of a directed edge colour it red, if at the tail, blue, and otherwise arbitrarily. A node is only forced to be both red and blue if there is a directed path of length $w=2$. Lemma 16 bounds the probability of this. □

4.5 Computing the Probability of Failure

We now determine the probability that our algorithm for generating a balanced allocation is not successful. The probability that the total same-player-vee graph is not 2-colorable is at most $\frac{16d^3}{\alpha^3}+\frac{8d^2}{\alpha^2}$. The probability that the implication graph contains a pair path of length three or more is at most $\frac{32d^3}{\alpha^2(\alpha-d^2)}$. Thus we get that the probability that the maximum overlap of the final pieces is more than 2 is at most $\frac{16d^3}{\alpha^3}+\frac{8d^2}{\alpha^2}+\frac{32d^3}{\alpha^2(\alpha-d^2)}$. By setting $d=2$, and then setting $\alpha\ge 10$ to be sufficiently large, one can make this probability arbitrarily small. Hence, the probability that our caking cutting algorithm is not at least 2α-fair is at most $O(1/\alpha^2)$.

4.6 Approximate Cuts with a Weak Adversary.

In this section, we show that even if the cut operations are only approximate, then approximate fairness is still achievable in $O(n)$ complexity against a weak adversary, which must specify the relative error without knowing the value of the cake specified in the cut.
$\boldsymbol{ACut_p(\epsilon,x_1,\beta)}$: This $1+\epsilon$ approximate cut query returns an $x_2\ge x_1$ such that the interval of cake $[x_1,x_2]$ has value approximately β according to player p's value function V_p. More precisely, x_2 satisfies $\frac{1}{1+\epsilon}V_p(x_1,x_2)\le\beta\le(1+\epsilon)V_p(x_1,x_2)$.

Adaptive Error: We say that $ACut_p(\epsilon, x_1, \beta)$ has an *adaptive error* if each operation the algorithm first provides x_1 and β and then the adversary, knowing the complete history, provides the worst case result x_2 within the stated error.
Non-Adaptive Error: We say that $ACut_p(\epsilon, x_1, \beta)$ has a *nonadaptive error* if each operation the algorithm first provides x_1 but not β. The weak adversary, knowing the complete history but not β, chooses a random variable E for the error with some distribution in the range $[\frac{1}{1+\epsilon}, 1+\epsilon]$. When the algorithm provides β, the operation $ACut_p(\epsilon, x_1, \beta)$ returns the random variable $x_2 = Cut_p(x_1, E \cdot \beta)$ such that $V_p(x_1, x_2) = E \cdot \beta$.

Theorem 17. *[4] If a protocol can only make $1+\epsilon$ approximate queries with an adaptive adversary, and c-fairness is required, then the complexity of any randomized protocol for cake cutting is $\Omega(n \log \frac{n}{c} / \log \frac{1}{\epsilon})$.*

Theorem 18. *If a protocol can only make $1+\epsilon$ approximate queries against a weak adversary, then there is a randomized protocol for cake cutting that achieves $O(1)$-fairness in $O(n)$ time.*

Proof. The algorithm as defined above chooses a random integer $i \in [0, \alpha n - 1]$ and cuts out a piece starting at $x_1 = Cut_p(0, \frac{i}{\alpha n})$ and ending at $x_2 = Cut_p(0, \frac{i+1}{\alpha n})$ or equivalently at $x_2 = Cut_p(x_1, \frac{1}{\alpha n})$. If the second cut is replaced with the cut $x_2 = ACut_p(\epsilon, x_1, \frac{1}{(1+\epsilon)\alpha n})$ even with adaptive error, then the algorithm does not change significantly. The piece returned is no wider so overlaps with other player's intervals are no more likely and the associated value, though perhaps a factor of $(1+\epsilon)^2$ more unfair, is still constant fair.

For the first cut $x_1 = Cut_p(0, \frac{i}{\alpha n})$, if the algorithm instead chooses a random real $i \in [0, \alpha n - 1]$ instead of a random integer, the algorithm does not change significantly. This then become a cut at a uniformly chosen random value $\beta = \frac{i}{\alpha n} \in [0, 1]$. If we replace this cut with an approximate cut with an non-adaptive adversary, it becomes a cut at value $\beta' = E\beta$. But because error E is a random variable is independent of β, β' is basically also a uniformly chosen random value $\beta' \in [0, 1]$. To see, this consider some fixed value $b \in [\epsilon, 1-\epsilon]$ not too close to the endpoints. We have

$$\Pr\left[\beta' \in [b, b+\delta b]\right] = \int_{e \in [\frac{1}{1+\epsilon}, 1+\epsilon]} \Pr\left[\beta \in [\frac{b}{e}, \frac{b+\delta b}{e}]\right] \cdot \Pr\left[E = e\right] \delta e$$

$$= \int_{e \in [\frac{1}{1+\epsilon}, 1+\epsilon]} \frac{\delta b}{e} \cdot \Pr\left[E = e\right] \delta e = \delta b \cdot \left[\int_{e \in [\frac{1}{1+\epsilon}, 1+\epsilon]} \frac{\Pr\left[E = e\right]}{e} \delta e\right].$$

This is a strange integration, but it is within $(1+\epsilon)$ of one and it is constant with respect to b. Hence, $\Pr\left[\beta' \in [b, b+\delta b]\right] \approx \delta b$, meaning that β' is uniformly chosen within $[\epsilon, 1-\epsilon]$. ☐

5 Conclusion

The are several lines of further inquiry. One could try to improve our Balanced Allocation Lemma by reducing d to 1, or by proving a high probability result. One could try to determine if linear complexity is obtainable for cake cutting if either exact fairness or contiguous portions were required. But perhaps most interesting is to see how other balanced allocation results in the literature extend to the unrelated machines case. The obvious first step would be to determine happens in the sequential/online case.

References

[1] Y. AZAR, (1998) On-line Load Balancing, *Online Algorithms - The State of the Art*, Springer, 178-195.

[2] Y. AZAR, A. BRODER, A. KARLIN, AND E. UPFAL, (2000) Balanced Allocations, SIAM Journal of Computing 29, 180–200.

[3] S.J. BRAMS AND A.D. TAYLOR (1996). *Fair Division – From cake cutting to dispute resolution*. Cambridge University Press, Cambridge.

[4] J. EDMONDS AND K. PRUHS, Cake cutting really isn't a piece of cake, *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'2006)*.

[5] S. EVEN AND A. PAZ (1984). A note on cake cutting. *Discrete Applied Mathematics 7*, 285–296.

[6] S.O. KRUMKE, M. LIPMANN, W. DE PAEPE, D. POENSGEN, J. RAMBAU, L. STOUGIE, AND G.J. WOEGINGER (2002). How to cut a cake almost fairly. *Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'2002)*, 263–264.

[7] M. MAGDON-ISMAIL, C. BUSCH, AND M.S. KRISHNAMOORTHY (2003). Cake cutting is not a piece of cake. *Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science (STACS'2003)*, LNCS 2607, Springer Verlag, 596–607.

[8] M. MITZENMACHER, A. RICHA, AND R. SITARAMAN 200 The power of two random choices: A survey of the techniques and results, *Handbook of Randomized Computing*, editors: P. Pardalos, S. Rajasekaran, and J. Rolim, Kluwer.

[9] J.M. ROBERTSON AND W.A. WEBB (1995). Approximating fair division with a limited number of cuts. *Journal of Combinatorial Theory, Series A 72*, 340–344.

[10] J.M. ROBERTSON AND W.A. WEBB (1998). *Cake-cutting algorithms: Be fair if you can*. A.K. Peters Ltd.

[11] J. SGALL AND G. J. WOEGINGER (2003). A lower bound for cake cutting. LNCS 2461, Springer Verlag, 896–901. *Proc. of the 11th Ann. European Symp. on Algorithms (ESA), Lecture Notes in Comput. Sci. 2832*, Springer, 459–469.

[12] H. STEINHAUS (1948). The problem of fair division. *Econometrica 16*, 101–104.

[13] G.J. WOEGINGER (2002). An approximation scheme for cake division with a linear number of cuts. *Proc. of the 10th Ann. European Symp. on Algorithms (ESA), Lecture Notes in Comput. Sci. 2461*, Springer, 896–901.