1 Gaussian Distribution Analysis

In this section, we will redo the proof that the disadvantaged person will have a higher expected talent then the advantaged person given the same performance score. However, this time the talent $T = \mathcal{N}(\mu_T, \sigma_T^2)$ and environment $E = \mathcal{N}(\mu_E, \sigma_E^2)$ distributions will be Gaussian distributions instead of normal. Because the sum of Gaussians is Gaussian, we know the performance score $X = T+E$ is also Gaussian $X = \mathcal{N}(\mu_T+\mu_E, \sigma_T^2+\sigma_E^2)$. Because the expectation of the sum $X = T+E$ is the sum expectation, we easily get $Exp(T|X = x) = x - Exp(E)$. What is fun is that we will used Bayes' rule to show that the conditional distribution $[T|X = T+E = x]$ is also Gaussian. Specifically the rule gives

$$Pr(T \in [t, t+\delta t] | X \in [x, x+\delta x]) \;=\; Pr(X \in [x, x+\delta x] | T \in [t, t+\delta t]) \times \frac{Pr(T \in [t, t+\delta t])}{Pr(X \in [x, x+\delta x])}$$

Conditioned on T having some fixed value of $t' \in [t, t+\delta t]$, we independently choose a value for E and then compute $X = t'+E$. Hence, we get $X \in [x, x+\delta x]$ iff $E \in [x-t', x-t'+\delta x]$. Because E, T, and E are normally distributed we get

$$Pr(E \in [x-t', x-t'+\delta x]) = \frac{1}{\sqrt{2\pi}\sigma_E} e^{-(e-\mu_E)^2/(2\sigma_E^2)} \times \delta x \text{ where } e = x-t,\ \mu_E = \mu_T+\mu_E,\ \sigma_E = \sigma_T^2+\sigma_E^2$$

$$Pr(T \in [t, t+\delta t]) = \frac{1}{\sqrt{2\pi}\sigma_T} e^{-(t-\mu_T)^2/(2\sigma_T^2)} \times \delta t$$

$$Pr(X \in [x, x+\delta x]) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-(x-\mu_X)^2/(2\sigma_X^2)} \times \delta x$$

Plugging these into Bayes' rule gives

$$Pr(T \in [t, t+\delta t] | X \in [x, x+\delta x]) = \frac{1}{\sqrt{2\pi}\sigma'_T} e^{-(t-\mu'_T)^2/(2\sigma'^2_T)} \times \delta t$$

where

$$\frac{(t-\mu'_T)^2}{2\sigma'^2_T} = \frac{(x-t-\mu_E)^2}{2\sigma_E^2} + \frac{(t-\mu_T)^2}{2\sigma_T^2} - \frac{(x-\mu_X)^2}{2(\sigma_T^2+\sigma_E^2)}$$

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Which can be written as:

$$\text{Pr(Y=y} \mid X_B - X_A = 0) = \frac{\sqrt{\sigma_T^2+\sigma_E^2}}{2\sqrt{\pi}\sigma_T\sigma_E} e^{-\frac{1}{4}\left(\frac{(y-K)^2}{\sigma_E^2}+\frac{y^2}{\sigma_T^2}-\frac{K^2}{\sigma_T^2+\sigma_E^2}\right)}$$

The above expression can be simplified to:

$$Pr(Y = y \mid X_B - X_A = 0) = \frac{1}{2\sqrt{\pi}\sigma_A} e^{-\frac{1}{2\sigma_A^2}\left(y+\frac{-K}{\sqrt{2}\sigma_E^2/\sigma_A^2}\right)^2}$$

where $\sigma_A = \frac{\sigma_T\sigma_E}{\sqrt{\sigma_T^2+\sigma_E^2}}$

Equation **??** could now be compared to the Probability Density Function of a Normal distribution with mean and variance:

$$\mu = \frac{-r}{\sqrt{2}(1+\frac{\sigma_E^2}{\sigma_T^2})} \text{ and } \sigma^2 = 2\sigma_A^2 = \frac{2\sigma_T^2\sigma_E^2}{\sqrt{\sigma_T^2+\sigma_E^2}}$$

Therefore the Distribution is equivalent to:

$$(T_B - T_A \mid X_B - X_A = 0) =$$

$$\mathrm{N}\Big(\mathrm{K}\frac{}{\sqrt{2}(1+\frac{\sigma_E^2}{\sigma_T^2}), \frac{2\sigma_T^2\sigma_E^2}{\sqrt{\sigma_T^2+\sigma_E^2}}}\Big)$$

Since r is negative, therefore the expected value(mean) of the above distribution positive. To conclude, if $t_a, \in T_A$, $t_b \in T_B$, $x_a \in X_A$ and $x_b \in X_B$, then

$$Exp(t_b - t_a | x_b - x_a = 0) = \frac{K}{\sqrt{2}(1+\frac{\sigma_E^2}{\sigma_T^2})}$$

We could equivalently say that $Exp[T \mid X = x \,\&\, G = B] > Exp[T \mid X = x \,\&\, G = A]$. Hence, we have shown that for Gaussian Distributions, the expected value of talents for the disadvantaged group is strictly expected to be greater than the advantaged group.