EECS/MATH 1019
Sections 5.1–5.2:
Mathematical Induction — Part II

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Principle of Mathematical Induction

Let $P(1), P(2), P(3), \ldots$ *be statements. Assume*
(a) $P(1)$ *is true, and*
(b) $(\forall k \in \mathbb{Z}^+)\,(P(k) \to P(k+1))$.
Then $P(n)$ *is true for every* $n \in \mathbb{Z}^+$.

Modified Principle of Mathematical Induction

Let $M \in \mathbb{Z}$, *and let* $P(M), P(M{+}1), P(M{+}2), \ldots$ *be statements. Assume*

(a') $P(M)$ *is true, and*
(b') $P(k) \to P(k+1)$ *for every* $k \in \mathbb{Z}$ *such that* $k \geq M$.
Then $P(n)$ *is true for every* $n \in \mathbb{Z}$ *such that* $n \geq M$.

Notes: (1) Assumption (b') says that $P(M) \to P(M{+}1)$ and $P(M{+}1) \to P(M{+}2)$ and $P(M{+}2) \to P(M{+}3)$ and $\ldots$.
(2) The usual Principle of Mathematical Induction is a special case of the Modified Principle of Mathematical Induction with $M = 1$.

Example 7 Prove that for every positive integer n,

$$\sum_{i=1}^{n} (i+1)\, 2^i = n\, 2^{n+1} .$$

That is, $2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1)\, 2^n = n\, 2^{n+1}$.

First, we'll do a quick check:

m	1	2	3
$(m+1)\, 2^m$	$2 \cdot 2^1 = 4$	$3 \cdot 2^2 = 12$	$4 \cdot 2^3 = 32$
$\sum_{i=1}^{m} (i+1)\, 2^i$	4	16	48
$m\, 2^{m+1}$	$1 \cdot 2^{1+1} = 4$	$2 \cdot 2^{2+1} = 16$	$3 \cdot 2^{3+1} = 48$

Looks okay so far! Now we need to prove that the equation is ALWAYS true.

Proof: For each positive integer n, let $P(n)$ be the statement $\sum_{i=1}^{n} (i+1)\, 2^i = n\, 2^{n+1}$.
The above table shows that $P(1)$, $P(2)$, and $P(3)$ are all true. We shall use mathematical induction to prove that $P(n)$ is true for EVERY positive integer n.

For each $n \in \mathbb{Z}^+$, $P(n)$ is the statement $\sum_{i=1}^{n} (i+1)\,2^i \;=\; n\,2^{n+1}$.

(That is, $2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1)\,2^n \;=\; n\,2^{n+1}$.)

Basis step: We have shown that $P(1)$ is true.

Inductive step: Let $k \in \mathbb{Z}^+$, and assume that $P(k)$ is true. We need to show that $P(k+1)$ must also be true (i.e., that $P(k) \to P(k+1)$). Write down what we want to do:

Show $2 \cdot 2 + 3 \cdot 2^2 + \cdots + (k+2)\,2^{k+1}$ (call this A)

equals $(k+1)\,2^{k+2}$ (call this B).

Important point: From $P(k)$, we know
$2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (k+1)\,2^k \;=\; k\,2^{k+1}$.
Now express A in a form that lets us leverage this fact:

$$
\begin{aligned}
A \;&=\; 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (k+1)\,2^k + (k+2)\,2^{k+1} \\
&=\; k\,2^{k+1} + (k+2)\,2^{k+1} \qquad (\text{by } P(k)) \\
&=\; (\,k + (k+2)\,)\,2^{k+1} \\
&=\; (2k+2)\,2^{k+1} \;=\; (k+1)(2)2^{k+1} \;=\; (k+1)\,2^{k+2} \;=\; B\,.
\end{aligned}
$$

Thus, we proved $A = B$. This completes the inductive step.

Summary:

For each $n \in \mathbb{Z}^+$, $P(n)$ is the statement $\sum_{i=1}^{n}(i+1)\,2^i \;=\; n\,2^{n+1}$.
We have proved $P(1)$ is true. (Basis step)
We have proved that $P(k) \to P(k+1)$ for every integer k such that $k \geq 1$.
Therefore, by mathematical induction, $P(n)$ is true for every integer n such that $n \geq 1$.
That is, $\sum_{i=1}^{n}(i+1)\,2^i \;=\; n\,2^{n+1}$ for every positive integer n.

Example 8 For which positive integers n is it true that $n! \geq \frac{1}{4}\,3^n$?

Recall $n! = n \times (n-1) \times \ldots \times 2 \times 1$.

n	1	2	3	4	5
$n!$	1	2	6	24	120
$\frac{1}{4}\,3^n$	$\frac{3}{4}$	$\frac{9}{4}$	$\frac{27}{4}$	$\frac{81}{4}$	$\frac{243}{4}$
$n! \geq \frac{1}{4}\,3^n$?	Yes	No	No	Yes	Yes

Let's try to prove that the inequality holds for all $n \geq 4$.

Let $P(n)$ be the statement $n! \geq \frac{1}{4}\,3^n$.

Basis step: We know that $P(4)$ is true.

Inductive step: Assume that k is an integer such that $k \geq 4$ and $P(k)$ is true. We want to prove that $P(k+1)$ is also true.

$$P(k): \quad k! \geq \frac{1}{4}\,3^k \qquad\qquad P(k+1): \quad (k+1)! \geq \frac{1}{4}\,3^{k+1}$$

We have assumed $P(k): \quad k! \geq \frac{1}{4}3^k$

We want to deduce $P(k+1): \quad (k+1)! \geq \frac{1}{4}3^{k+1}$

Useful fact: $(k+1)! = (k+1) \times (k!)$. So we obtain

$$
\begin{aligned}
(k+1)! &= (k+1) \times k! \\
&\geq (k+1) \times \frac{1}{4}3^k \qquad \text{(by } P(k)\text{)} \\
&\geq 3 \times \frac{1}{4} \times 3^k \qquad \text{(because } k \geq 2 \text{ (in fact, } k \geq 4\text{))} \\
&= \frac{1}{4}3^{k+1}.
\end{aligned}
$$

This proves $P(k+1)$.

Summary: We proved that $P(4)$ is true, and that $P(k) \rightarrow P(k+1)$ for all integers $k \geq 4$.

We conclude that $P(n)$ is true for all $n \geq 4$.

Section 5.2: Strong Induction

Let $P(1), P(2), P(3), \ldots$ *be statements. Assume*

(a') $P(1)$ is true, and
(b'') $[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ for every $k \in \mathbb{Z}^+$.

Then $P(n)$ is true for every $n \in \mathbb{Z}^+$.

(Note: Assumption (b'') says that $P(1) \rightarrow P(2)$ and $[P(1) \wedge P(2)] \rightarrow P(3)$ and $[P(1) \wedge P(2) \wedge P(3)] \rightarrow P(4)$ and . . . .)

Like ordinary induction, we can start at another integer besides 1. For example, we can use the following form:

Let $P(0), P(1), P(2), \ldots$ *be statements. Assume*

(a') $P(0)$ is true, and
(b'') $[P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ for every $k \in \mathbb{N}$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Definition: A prime is an integer greater than 1 whose only factors are 1 and itself.
E.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 are all the primes less than 30.

Example 9: (p. 357) Use strong induction to prove that every integer greater than 1 is either a prime or a product of primes.

E.g. $26 = 2 \times 13 \qquad 60 = 2 \times 2 \times 3 \times 5$

Let $P(n)$ be the statement "n is a prime or a product of primes."
We shall prove that $P(n)$ is true for every integer $n \geq 2$.

Basis step: $n = 2$: $P(2)$ is true because 2 is a prime.

Inductive step: We must show that for every integer $k \geq 2$,
$[P(2) \wedge P(3) \wedge \ldots \wedge P(k)] \to P(k+1)$.
So assume $k \geq 2$ and $[P(2) \wedge P(3) \wedge \ldots \wedge P(k)]$ is true.
For $k+1$, we consider two cases: $k+1$ is either a prime or not a prime.

Let $P(n)$ be the statement "n is a prime or a product of primes."
We want to prove that $P(n)$ is true for every integer $n \geq 2$.
Assume $k \geq 2$ and $[P(2) \wedge P(3) \wedge \ldots \wedge P(k)]$ is true.

Case I: $k+1$ is a prime: In this case, $P(k+1)$ is obviously true.

Case II: $k+1$ is not a prime: Then $k+1$ has a divisor d besides 1 and $k+1$. Thus $k+1 = dw$ for some integer w, and $1 < d < k+1$ (since $d \neq 1$ and $d \neq k+1$). Also $1 < w < k+1$.
Therefore $2 \leq d \leq k$ and $2 \leq w \leq k$. So, by the inductive assumption that $[P(2) \wedge P(3) \wedge \ldots \wedge P(k)]$ is true, we see that both $P(d)$ and $P(w)$ are true.
That is, d is a prime or a product of primes, and w is a prime or a product of primes.
Therefore dw is a product of primes. Since $k+1 = dw$, this proves that $k+1$ is a product of primes. So $P(k+1)$ is true in Case II.

Since both cases have been checked, we conclude that $[P(2) \wedge P(3) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$. This is true for every integer $k \geq 2$, so have proved the inductive step (b").

Therefore, $P(n)$ is true for every $n \geq 2$ by Strong Induction.

We can also prove the preceding exercise (and other induction problems) by appealing to the **Well-Ordering Property of the positive integers** (Section 5.2.5). Here is how we can do it.

The Well-Ordering Property says that *every nonempty subset of the positive integers has a smallest element.* (Section 5.2.5)

Let $P(n)$ be the statement "n is a prime or a product of primes."

Consider the set $S = \{n \in \mathbb{Z}^+ \mid n \geq 2 \text{ and } P(n) \text{ is False}\}$.
Assume S is not empty. (We will use this to get a contradiction.)

By the Well-Ordering Property, S has a smallest element.

Let t be the smallest element of S.

Recap: $P(n)$ says "n is a prime or a product of primes."

Consider the set $S = \{n \in \mathbb{Z}^+ \mid n \geq 2 \text{ and } P(n) \text{ is False}\}$.
Assume S is not empty. (We will use this to get a contradiction.)
Let t be the smallest element of S. (Well-Ordering)

In particular, since $t \in S$, we know that $P(t)$ is False.
Therefore t is not a prime. Therefore (as we argued before) we can write $t = dw$ where $2 \leq d < t$ and $2 \leq w < t$.

Since d and w are both LESS than t, and t is the SMALLEST element of S, neither d nor w is an element of S.
Therefore $P(d)$ and $P(w)$ are not False; that is, they are True.

So, as before: d is a prime or a product of primes, and w is a prime or a product of primes. Therefore dw is a product of primes. Since $t = dw$, this proves that t is a product of primes. Thus $P(t)$ is True. This CONTRADICTS the fact that $P(t)$ is False.

We conclude that S must be empty.
That is, $P(n)$ is True whenever $n \geq 2$.

Example 10: A Recursive Sequence Let $a_1, a_2, a_3, \ldots$ be the sequence of real numbers defined *recursively* by

$$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+1} = a_n + 2a_{n-1} \quad \text{for } n = 2, 3, 4, \ldots$$

Then $a_3 = a_2 + 2a_1 = 5 + 2(1) = 7$,

$a_4 =$

Poll: What is the value of a_4?

(A) 12
(B) 15
(C) 17
(D) 19

Answer: (C)

Example 10: A Recursive Sequence Let $a_1, a_2, a_3, \ldots$ be the sequence of real numbers defined *recursively* by

$$a_1 = 1, \quad a_2 = 5, \qquad \text{and} \qquad a_{n+1} = a_n + 2a_{n-1} \quad \text{for } n = 2, 3, 4, \ldots$$

Then
$$\begin{aligned} a_3 &= a_2 + 2a_1 = 5 + 2(1) = 7, \\ a_4 &= a_3 + 2a_2 = 7 + 2(5) = 17, \\ a_5 &= a_4 + 2a_3 = 17 + 2(7) = 31, \\ a_6 &= a_5 + 2a_4 = 31 + 2(17) = 65. \\ a_7 &= a_6 + 2a_5 = 65 + 2(31) = 127. \end{aligned}$$

Compare with the sequence 2^n: 2, 4, 8, 16, 32, 64, 128, ...

This suggests the formula $a_n = 2^n + (-1)^n$.

Mathematical induction is ideally suited to handling recursively defined quantities.

Let $P(n)$ be the statement $a_n = 2^n + (-1)^n$.

$a_1 = 1$, $a_2 = 5$, and $a_{n+1} = a_n + 2a_{n-1}$ for $n = 2, 3, 4, \dots$

Let $P(n)$ be the statement $a_n = 2^n + (-1)^n$.

We have checked that $P(n)$ is true for $n = 1, 2, 3, \dots, 7$.

The statement $P(k+1)$ is $a_{k+1} = 2^{k+1} + (-1)^{k+1}$.

To prove $P(k+1)$, we would need to use the definition $a_{k+1} = a_k + 2a_{k-1}$.

To use this, we would like to use formulas for a_k and a_{k-1}.

That is, we would like to know that $P(k)$ and $P(k-1)$ are both true. Then we would know $a_k = 2^k + (-1)^k$ and $a_{k-1} = 2^{k-1} + (-1)^{k-1}$. This would lead to

$$
\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= \left(2^k + (-1)^k\right) + 2\left(2^{k-1} + (-1)^{k-1}\right) \\
&= 2^k + 2(2^{k-1}) + (-1)^k + 2(-1)^{-1}(-1)^k \\
&= 2^k + 2^k + (-1)^k - 2(-1)^k \\
&= 2(2^k) - (-1)^k \qquad = 2^{k+1} + (-1)^{k+1}.
\end{aligned}
$$

This shows that $P(k+1)$ is true.

$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+1} = a_n + 2a_{n-1} \quad \text{for } n = 2, 3, 4, \ldots$

Here is the formal proof that $a_n = 2^n + (-1)^n$ for every $n \in \mathbb{Z}^+$.

Proof: Let $P(n)$ be the statement $a_n = 2^n + (-1)^n$.
Since $2^1 + (-1) = 1$ and $2^2 + (-1)^2 = 5$, we see directly that $P(1)$ and $P(2)$ are true. (In particular, $P(1) \to P(2)$ is true.)
Let $k \geq 2$ and assume that $P(1), P(2), \ldots, P(k-1)$, and $P(k)$ are all true. Then

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} \\ &= \left(2^k + (-1)^k\right) + 2\left(2^{k-1} + (-1)^{k-1}\right) \\ &\qquad\qquad\qquad \text{(using } P(k) \text{ and } P(k-1)\text{)} \\ &= 2^k + 2^k + (-1)^k - 2(-1)^k \\ &= 2(2^k) - (-1)^k \qquad = 2^{k+1} + (-1)^{k+1}. \end{aligned}$$

This shows that $P(k+1)$ is true.
We know that $P(1)$ is true, and
$[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \to P(k+1)$ for all $k \in \mathbb{Z}^+$.
Hence $P(n)$ is true for every $n \in \mathbb{Z}^+$ by Strong Induction. Q.E.D.

Next class: Read Section 5.3. (Subsection 5.3.5 is optional.)

Homework updates:

- Homework assignment 5 (in Connect) is due Sunday March 5.
- Problem Set B is also due Sunday March 5, to be submitted via Crowdmark.