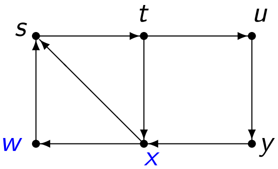
EECS/MATH 1019
Section 10.4:
Paths and Connectivity (concluded)
Section 10.5:
Euler and Hamilton Circuits and Paths

April 4, 2023

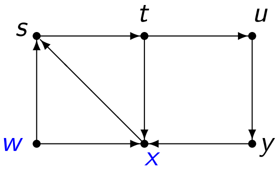
Sec. 10.4.5: Connectivity in Digraphs

The idea of connectivity is that it is possible to get from any vertex to any other vertex by a path.

C



D



The only difference between C and D is the direction of the edge between w and x. What does that do to the "connectivity"?

Definition: A digraph is strongly connected if there are paths from every vertex to every other vertex.

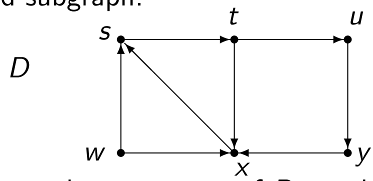
E.g. C is strongly connected but D is not.

Both C and D have the same underlying undirected graph, which is the graph obtained by replacing each directed edge with an undirected edge.

Definition: A digraph is weakly connected if its underlying undirected graph is connected.

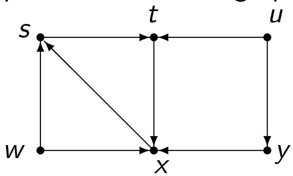
E.g. C and D are both weakly connected.

A strongly connected component of a digraph G is a strongly connected subgraph of G that is not contained in any larger strongly connected subgraph.



The strongly connected components of D are the single vertex w and the subgraph $D - w$.

Poll: If we reverse the edge from t to u, how many strongly connected components are in the graph?

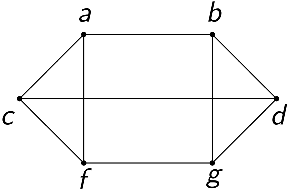


(A) 2
(B) 3
(C) 4
(D) 5

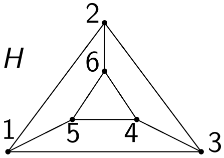
Answer: ??

Sec. 10.4.6: Paths, Circuits, and Isomorphisms

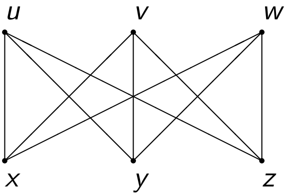
G



H



J



Poll:
Which of these graphs are isomorphic?

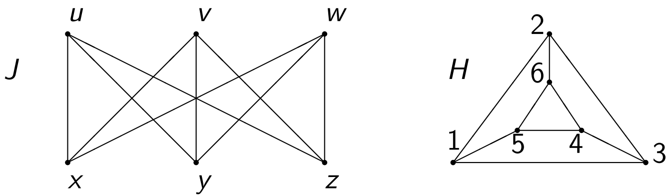
(A) G and H (B) G and J (C) H and J

(D) All three are isomorphic to one other

(E) No two of them are isomorphic

<u>Answer:</u> ??

Are we sure that J is not isomorphic to (G or H)?



We are sure because H has a circuit of length 3, while J does not.

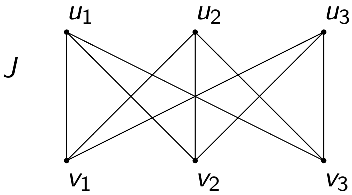
Observe that if f is and isomorphism from H to J, then vertices $f(1)$, $f(2)$, and $f(3)$ must all be adjacent to each other in J. Thus they would lie on a circuit of length 3 in J.

An analogous statement is true for any length $n \geq 3$:

If G and J are isomorphic graphs, and G has a circuit of length n, then J has a circuit of length n.

(You can omit Section 10.4.7. It is interesting, but requires knowledge of matrix multiplication.)

Remark: We observed that this graph J has no circuit of length 3.



Does J have a circuit of length 5? (It doesn't have to be simple)

Notice that J is bipartite, with bipartition $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. In every circuit (or path) of J, the vertices must alternate between the u's and the v's.

For a cycle of length 5, the vertex sequence would be $x_0, x_1, x_2, x_3, x_4, x_5$ with $x_0 = x_5$.

If x_0 is a u vertex, then x_1 is a v, and x_2 is a u,... and x_5 is a v.

But this can't happen, because $x_5 = x_0$ and x_0 is a u.

The same problem arises if x_0 is a v.

Conclusion: There cannot be a circuit of length 5 in J.

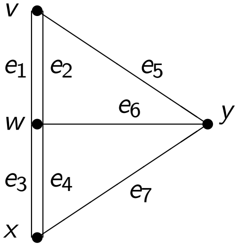
More generally:

Every circuit has even length in any bipartite graph.

Puzzle: The Seven Bridges of Königsberg

The Prussian town of Königsberg had two large islands in its river, with seven bridges connecting the islands and the river banks.
Problem: Is it possible to go for a walk, crossing each bridge exactly once, and end up where you started? If not, why not? See https://mathworld.wolfram.com/KoenigsbergBridgeProblem.html

Here is an equivalent question in terms of a (multi)graph. Think of each land mass (the two islands and the two banks) as a vertex, and each bridge as an edge.



River banks: v and x

Islands: w and y

Bridges: $e_1, \ldots, e_7$

The question is: Is there a circuit in this graph that uses each edge exactly once? I.e., is there a <u>simple</u> circuit that uses each edge?

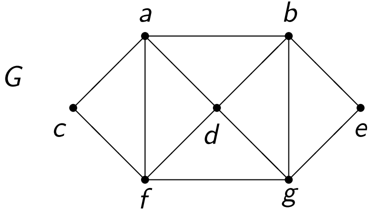
This problem was studied and solved in 1736 by Leonhard Euler (pronounced “Oiler,” like “Freud” or “Deutsche”)

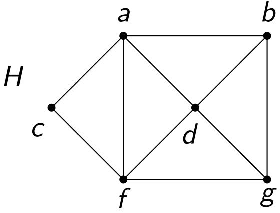
Definition: An Euler circuit in a graph is a simple circuit that contains every edge of the graph. An Euler path in a graph is a simple path that contains every edge of the graph.

Euler circuits and paths have numerous applications. For example, for a network of streets in a town, they can be used to plan routes for letter carriers, or sidewalk snow plows, or parking enforcement officers checking parking meters on foot.

An Euler circuit is desired if the worker must return to a central depot; an Euler path is sufficient if the worker can do whatever they wish after they finish.

The definition also applies to directed graphs. Directed edges are necessary when street directions are relevant: e.g. garbage pickup, or clearing streets of snow.





Do these graphs have Euler circuits? Euler paths?

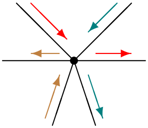
G has an Euler circuit, e.g. $c\,a\,b\,e\,g\,f\,d\,g\,b\,d\,a\,f\,c$
This is also an Euler path.

Poll: Which is true about *H*?
(A) *H* has an Euler circuit and an Euler path
(B) *H* has an Euler circuit but no Euler path
(C) *H* has an Euler path but no Euler circuit
(D) *H* has no Euler circuit and no Euler path

Answer: ???

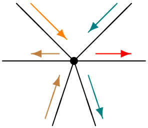
Observation: If a graph G has an Euler circuit, then every vertex of G has even degree.

Why is this true? Assume that G has an Euler circuit. Let v be any vertex of G.
As we follow the Euler circuit around G, we enter v the same number of times that we leave v. Each time we enter and each time we leave, we must use a different edge (since an Euler circuit is simple).



And so v ends up with an even degree.
This works even if there is a loop at v (since the loop counts for two edges in the degree.)

One special situation: If the circuit starts and ends at v, then the picture is slightly different:



And v still ends up with an even degree!

This is how the observation is proved.

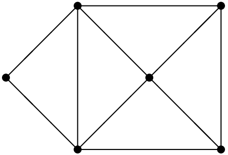
We have proved that **if** G has an Euler circuit, **then** every vertex has even degree.

It does not logically follow from this that if every vertex has even degree in a graph, then the graph has an Euler circuit. However, it turns out that this is essentially true!

Theorem 1 (p. 731) Let G be a multigraph with at least two vertices. Then G has an Euler circuit *if and only if* every vertex of G has even degree and G is connected.

See text for the proof.

H

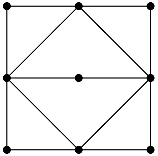


What does Theorem 1 say about H? Since H has two vertices of degree 3, it cannot have an Euler circuit.

We saw that H has an Euler path. What is special about a graph with an Euler path but no Euler circuit?

Theorem 2 (p. 733) Let G be a connected multigraph with at least two vertices. Then G has an Euler path but no Euler circuit *if and only if* exactly two vertices of G have odd degree.

Θ

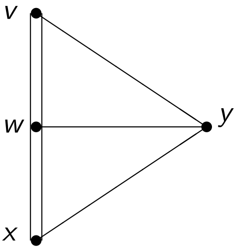


What does Theorem 2 say about Θ?
Two vertices have degree 5, and the rest have even degrees.
So it has an Euler path
(but not an Euler circuit).

How do we find the Euler path? The reasoning we sketched for the proof of Theorem 1 suggests that any Euler path must start at one of the two vertices of odd degree, and must end at the other one.

Knowing this, it is pretty easy to find Euler paths. Try some examples — with a bit of practice, it is hard *not* to find them when they are there!

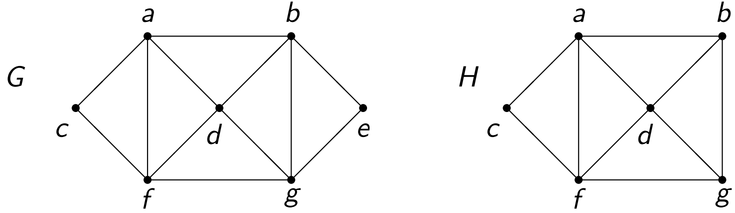
Meanwhile, back in Königsberg:



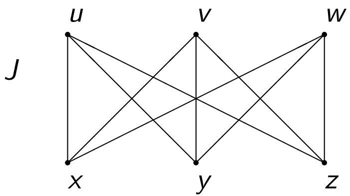
All four vertices have odd degree, so not only does this graph have no Euler circuit, but it does not even have an Euler path.

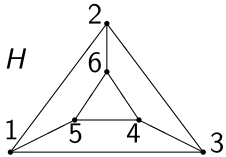
10.5.3: Hamilton Circuits and Paths

Definition: A Hamilton circuit in a graph is a circuit that contains every vertex of the graph exactly once (except the first occurs twice, because it is also the last vertex).
A Hamilton path in a graph is a path that contains every vertex of the graph exactly once.



Do these graphs have Hamilton circuits?
There is a Hamilton circuit in G: for example, $c\ a\ b\ e\ g\ d\ f\ c$.
There is also a Hamilton circuit in H: for example, $d\ b\ g\ f\ c\ a\ d$.





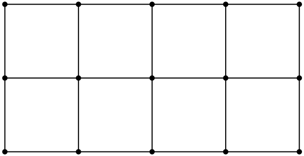
Poll: Which of these graphs have Hamilton circuits?

(A) J only
(B) H only
(C) both
(D) neither

Answer: ??

Poll: Does this graph have a Hamilton circuit?

(A) Yes (B) No



Answer: ??

Please visit the course evaluation web site:
https://courseevaluations.yorku.ca

For the final class: Read Section 11.1.

The final Connect assignment is due Monday April 10.

Exam information will be on the eClass page very soon.
I will also announce my office hours for the exam period there.