EECS/MATH 1019
Sections 8.2: Recurrence Relations (concluded)
Section 9.1: Relations

March 9, 2023

Recap: Section 8.2 gives some recipes for finding solutions to some special kinds of recurrence relations, namely linear recurrence relations with constant coefficients. Some examples are

$$f_n \;=\; f_{n-1} + f_{n-2}$$

$$\text{and} \qquad g_n \;=\; 6\,g_{n-1} - 7\,g_{n-2} + 3.2\,g_{n-4}$$

$$\text{and} \qquad h_n \;=\; 1.6\,h_{n-1} + \,h_{n-2} - 13\,h_{n-3} \,+\, n^2 - 5\,.$$

These are all of the form

$$a_n \;=\; c_1\,a_{n-1} \,+\, c_2\,a_{n-2} \,+\, \cdots \,+\, c_k\,a_{n-k} \,+\, F(n)$$

where F is any function, the "coefficients" $c_1, \ldots, c_k$ are real constants (i.e., cannot depend on n), and k is the "degree" (assuming $c_k \neq 0$; note that the other c_i's could be 0, such as c_3 in the recursion for g_n).
The relation is homogeneous if $F(n) = 0$ for every n. (As for $\{f_n\}$ and $\{g_n\}$ above.) Otherwise it is nonhomogeneous (as for $\{h_n\}$).

8.2.2: Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

That is, we focus on relations of the form

$$a_n \;=\; c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_k\, a_{n-k}\,. \qquad (*)$$

<u>Observation:</u> Let r be a real number (constant). Then the sequence defined by $a_n = r^n$ satisfies $(*)$ if and only if r satisfies the equation

$$r^k \,-\, c_1\, r^{k-1} \,-\, c_2\, r^{k-2} \,-\, \cdots \,-\, c_{k-1} r \,-\, c_k \;=\; 0 \qquad (CE)$$

called the characteristic equation (CE) of the recurrence relation $(*)$. The solutions of (CE) are the characteristic roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Assume that the characteristic equation $r^k - c_1\, r^{k-1} - c_2\, r^{k-2} - \ldots - c_{k-1}\, r - c_k = 0$ has k different roots $r_1, r_2, \ldots, r_k$. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots + c_k\, a_{n-k}$ are precisely the sequences $a_n = b_1\, r_1^n + \ldots + b_k\, r_k^n$, where $b_1, \ldots, b_k$ are constants.

Special case: When $k = 2$, the recurrence relation is $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$, and its associated characteristic equation is $r^2 - c_1\, r - c_2 = 0$.

In this case, Theorem 3 specializes to the following:

Theorem 1: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1\, r - c_2 = 0$ has two different roots r_1 and r_2. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ are precisely the sequences $a_n = b\, r_1^n + d\, r_2^n$, where b and d are constants.

Theorem 4 deals with the general situation of multiple roots. For example, if the characteristic equation (CE) consists of a polynomial of degree 6 and factors to

$$(r-2)^3(r-3.2)^2(r+4) \;=\; 0\,,$$

then the characteristic roots are 2, 2, 2, 3.2, 3.2, and -4.

We say that there are 6 real roots but only 3 <u>distinct</u> real roots. We can label them $r_1 = 2$, $r_2 = 3.2$, and $r_3 = -4$.

For each i, we write m_i to denote the <u>multiplicity</u> of the root r_i, which is the number of times that r_i occurs as a root in the characteristic equation.
In this example, $m_1 = 3$, $m_2 = 2$, and $m_3 = 1$.

Theorem 4: Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Assume that the characteristic equation $r^k - c_1\, r^{k-1} - c_2\, r^{k-2} - \ldots - c_{k-1}\, r - c_k = 0$ has k real roots, but only t distinct roots $r_1, r_2, \ldots, r_t$. For $i = 1, \ldots t$, let m_i be the multiplicity of the root r_i.
(*Note:* $\sum_{i=1}^{t} m_i = k$ *because there are* k *real roots.*)
Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots + c_k\, a_{n-k}$ are precisely the sequences $a_n = p_i(n)\, r_1^n + \ldots + p_t(n)\, r_t^n$, where p_i is a polynomial of degree at most $m_i - 1$.

Special case: For $k = 2$, the recurrence relation is $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$. If its associated characteristic equation $r^2 - c_1\, r - c_2 = 0$ has only one real root, then in Theorem 4 we have $t = 1$ and $m_1 = 2$, and we get the following:

Theorem 2: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1\, r - c_2 = 0$ has only one real root r_1. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ are precisely the sequences $a_n = (b\, n + d)\, r_1^n$, where b and d are constants.

8.2.3: Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Now we focus on relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (*_F)$$

where the function F is not identically 0.

E.g. (*i*) $s_n = 1.05\, s_{n-1} - 70$ (recall Example C)

(*ii*) $a_n = 5\, a_{n-1} - 6\, a_{n-2} + n^2$

Removing the $F(n)$ term from $(*_F)$ produces the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (*_H)$$

E.g. for (*i*), we get $s_n = 1.05\, s_{n-1}$ $(*_H)$

For (*ii*), we get $a_n = 5\, a_{n-1} - 6\, a_{n-2}$ $(*_H)$.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (*_F)$$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (*_H)$$

Theorem 5: Let $\{a_n^{(p)}\}$ be any one particular solution of $(*_F)$. Then *every* solution of $(*_F)$ is of the form $a_n^{(p)} + a_n^{(h)}$ where $a_n^{(h)}$ is a solution of $(*_H)$.

E.g. In Example (C), we found the the value s_n of Sandra's investments after n years (in thousands of dollars) satisfies

$$s_0 = 500 \quad \text{and} \quad s_n = 1.05\, s_{n-1} - 70 \quad (*_F) \quad \text{for } n \geq 1.$$

The associated homogeneous recurrence $(*_H)$ is $s_n = 1.05\, s_{n-1}$, and its most general solution is $s_n^{(h)} = b(1.05)^n$ (for any real constant b).

Now we need to find just one particular solution to $(*_F)$. In the previous class, we made an educated guess. But more generally, we can use Theorem 6 instead of guessing.

Theorem 6: Consider the recurrence relation

$$a_n \;=\; c_1\, a_{n-1} \;+\; c_2\, a_{n-2} \;+\; \cdots \;+\; c_k\, a_{n-k} + F(n) \qquad\qquad (*_F)$$

where F has the specific form

$$F(n) \;=\; (b_J n^J + b_{J-1} n^{J-1} + \ldots + b_1 n + b_0)\, s^n$$

for $b_J, \ldots, b_0, s \in \mathbb{R}$ (that is, a polynomial of degree J in n, times a constant to the power n). Then there is a particular solution of $(*_F)$ of the form

$$a_n^{(p)} \;=\; n^M (d_J n^J + d_{J-1} n^{J-1} + \ldots + d_1 n + d_0)\, s^n$$

where $d_J, \ldots, d_0 \in \mathbb{R}$ and

$$M \;=\; \begin{cases} m_i & \text{if } s \text{ equals the characteristic root } r_i \\ 0 & \text{if } s \text{ is not a characteristic root} \end{cases}$$

Theorem 6: (rewritten) Consider the recurrence relation

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_k\, a_{n-k} + F(n) \qquad (*_F)$$

$$\text{where} \quad F(n) = (b_J n^J + b_{J-1} n^{J-1} + \ldots + b_1 n + b_0)\, s^n$$

for $b_J, \ldots, b_0, s \in \mathbb{R}$. Then $(*_F)$ has a particular solution

$$a_n^{(p)} = n^M (d_J n^J + d_{J-1} n^{J-1} + \ldots + d_1 n + d_0)\, s^n, \quad \text{where}$$

$$d_J, \ldots, d_0 \in \mathbb{R}, \text{ and } \quad M = \begin{cases} m_i & \text{if } s = r_i \text{ (a characteristic root)} \\ 0 & \text{if } s \text{ is not a characteristic root} \end{cases}$$

For $s_n = 1.05\, s_{n-1} - 70$ as our $(*_F)$, we have
$F(n) = -70 = (-70)\, 1^n$ (notice $s = 1$ and $J = 0$).
Since 1 is not a characteristic root of $r - 1.05 = 0$, we are in the case $M = 0$.
So there is a particular solution $a_n^{(p)} = n^0 (d_0) 1^n$, i.e. $a_n^{(p)} = d_0$ (a constant solution).

Solve $d_0 = 1.05\, d_0 - 70$ to get $d_0 = \frac{70}{0.05} = 1400$, i.e. $a_n^{(p)} = 1400$.

$$
\begin{aligned}
s_n &= 1.05\, s_{n-1} - 70 \quad (n \geq 1) && (*_F) \\
s_n &= 1.05\, s_{n-1} \quad (n \geq 1) && (*_H)
\end{aligned}
$$

Thus we have found a particular solution for $(*_F)$, $a_n^{(p)} = 1400$; and we know the general solution for $(*_H)$, $a_n^{(h)} = b(1.05)^n$ (for any $b \in \mathbb{R}$).

So by Theorem 5, the general solution for $(*_F)$ is

$$s_n^{(p)} + s_n^{(h)} = 1400 + b(1.05)^n.$$

<u>Now</u> we bring in the initial condition to find out what b is.

We were told that $s_0 = 500$, so we have

$$500 = s_0 = 1400 + b\,(1.05)^0 = 1400 + b\,.$$

Therefore $b = -900$, and the solution for the complete problem is $s_n = 1400 - 900\,(1.05)^n \quad (n \geq 0)$.

Next, consider the example

$$(ii) \quad a_n \;=\; 5\,a_{n-1} \;-\; 6\,a_{n-2} \;+\; n^2 \qquad\qquad (*_F)$$

We examined the associated homogeneous recurrence relation

$$a_n \;=\; 5\,a_{n-1} \;-\; 6\,a_{n-2} \qquad\qquad (*_H)$$

in Example D in our previous class. The roots of the characteristic equation are 3 and 2, and the general solution of $(*_H)$ is

$$a_n^{(h)} \;=\; b\,3^n + d\,2^n \qquad \text{for any real } b \text{ and } d.$$

Theorem 6 helps us find a particular solution of $(*_F)$.

Theorem 6: Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (*_F)$$

$$\text{where} \quad F(n) = (b_J n^J + b_{J-1} n^{J-1} + \ldots + b_1 n + b_0) s^n$$

for $b_J, \ldots, b_0, s \in \mathbb{R}$. Then $(*_F)$ has a particular solution

$$a_n^{(p)} = n^M (d_J n^J + d_{J-1} n^{J-1} + \ldots + d_1 n + d_0) s^n, \quad \text{where}$$

$$d_J, \ldots, d_0 \in \mathbb{R}, \text{ and } \quad M = \begin{cases} m_i & \text{if } s = r_i \text{ (a characteristic root)} \\ 0 & \text{if } s \text{ is not a characteristic root} \end{cases}$$

For our example $a_n = 5 a_{n-1} - 6 a_{n-2} + n^2$, we have
$F(n) = n^2 = (1 \cdot n^2 + 0n + 0) 1^n$ (notice $s = 1$).
The characteristic roots are 3 and 2, so we are in the case $M = 0$.
So the form of the particular solution is

$$a_n^{(p)} = n^0 (d_2 n^2 + d_1 n + d_0) 1^n = d_2 n^2 + d_1 n + d_0.$$

Now what? Here is a rapid description of how to find d_0, d_1, and d_2, just to show you that it can be done if you really need to!

The form of the particular solution is

$$a_n^{(p)} = n^0(d_2\, n^2 + d_1\, n + d_0)\, 1^n = d_2\, n^2 + d_1\, n + d_0.$$

We can find the values of d_2, d_1, and d_0 by plugging into the recurrence equation $a_n = 5\, a_{n-1} - 6\, a_{n-2} + n^2$:

$$\begin{aligned} d_2\, n^2 + d_1\, n + d_0 \quad &= \quad 5[d_2\,(n-1)^2 + d_1\,(n-1) + d_0] \\ &\qquad -6[d_2\,(n-2)^2 + d_1\,(n-2) + d_0] \;+\; n^2\,. \end{aligned}$$

<u>Method 1</u>: Plug in some values of n, say $n = 0, 1, 2$:

$$\begin{aligned} n = 0: &\qquad 0 + 0 + d_0 \;=\; 5[d_2(-1)^2 + d_1(-1) + d_0] \\ &\qquad\qquad -6[d_2(-2)^2 + d_1(-2) + d_0] \;+\; 0^2 \\ \text{that is,} &\qquad d_0 = 5[d_2 - d_1 + d_0] - 6[4d_2 - 2d_1 + d_0] \\ n = 1: &\qquad d_2 + d_1 + d_0 \;=\; 5[d_2(0) + d_1(0) + d_0] \\ &\qquad\qquad -6[d_2(-1)^2 + d_1(-1) + d_0] \;+\; 1^2 \\ n = 2: &\qquad 4d_2 + 2d_1 + d_0 \;=\; 5[d_2(1)^2 + d_1(1) + d_0] \\ &\qquad\qquad -6[d_2(0)^2 + d_1(0) + d_0] \;+\; 2^2 \end{aligned}$$

This gives three linear equations in the three unknowns, which can be solved for d_0, d_1, and d_2.

$$
\begin{aligned}
d_2\, n^2 + d_1\, n + d_0 &= 5[d_2\,(n-1)^2 + d_1\,(n-1) + d_0] \\
&\qquad -6[d_2\,(n-2)^2 + d_1\,(n-2) + d_0] \;+\; n^2\,.
\end{aligned}
$$

Method 2: Expand the expression and match coefficients of n^2, n^1, and n^0.

$$
\begin{aligned}
d_2\, n^2 + d_1\, n + d_0 &= 5[d_2\,(n^2 - 2n + 1) + d_1\,(n-1) + d_0] \\
&\qquad -6[d_2\,(n^2 - 4n + 4) + d_1\,(n-2) + d_0] \;+\; n^2 \\
&= n^2[5d_2 - 6d_2 + 1] + \\
&\qquad n\,[5(-2d_2 + d_1) - 6((-4)d_2 + d_1)] + \\
&\qquad 5[d_2 - d_1 + d_0] - 6[(-4)d_2 - 2d_1 + d_0]\,.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad d_2 &= 5d_2 - 6d_2 + 1 \\
d_1 &= 5(-2d_2 + d_1) - 6((-4)d_2 + d_1) \\
d_0 &= 5[d_2 - d_1 + d_0] - 6[(-4)d_2 - 2d_1 + d_0]\,.
\end{aligned}
$$

Again, we can solve these equations for d_0, d_1, and d_2.

After some coffee, we get $d_2 = \frac{1}{2}$, $d_1 = \frac{7}{2}$, and $d_0 = \frac{15}{2}$.
So our particular solution is

$$a_n^{(p)} \ = \ d_2\, n^2 + d_1\, n + d_0 \ = \ \frac{1}{2}\, n^2 + \frac{7}{2}\, n + \frac{15}{2}\,.$$

Note that the initial conditions play no role in determining the d_i coefficients of the particular solution.
So the general solution of $a_n \ = \ 5\, a_{n-1} - 6\, a_{n-2} + n^2$ is

$$a_n \ = \ a_n^{(h)} + a_n^{(p)} \ = \ b\, 3^n + d\, 2^n \ + \ \frac{1}{2}\, n^2 + \frac{7}{2}\, n + \frac{15}{2}.$$

Now we can use the initial conditions (a_0 and a_1) to evaluate b and d via a system of two linear equations in the unknowns b and d.

Theorem 6: Consider the recurrence relation

$$a_n \;=\; c_1\, a_{n-1} \;+\; c_2\, a_{n-2} \;+\; \cdots \;+\; c_k\, a_{n-k} + F(n) \qquad (*_F)$$

$$\text{where} \quad F(n) \;=\; (b_J n^J + b_{J-1} n^{J-1} + \ldots + b_1 n + b_0)\, s^n$$

for $b_J, \ldots, b_0, s \in \mathbb{R}$. Then $(*_F)$ has a particular solution

$$a_n^{(p)} \;=\; n^M (d_J n^J + d_{J-1} n^{J-1} + \ldots + d_1 n + d_0)\, s^n, \quad \text{where}$$

$d_J, \ldots, d_0 \in \mathbb{R}$, and $M \;=\; \begin{cases} m_i & \text{if } s = r_i \text{ (a characteristic root)} \\ 0 & \text{if } s \text{ is not a characteristic root} \end{cases}$

For a slightly different example, if the recurrence relation is

$$a_n \;=\; 5\, a_{n-1} - 6\, a_{n-2} + n^2\, 2^n,$$

then we have $F(n) = n^2\, 2^n \;=\; (1 \cdot n^2 + 0n + 0)\, 2^n$.

Here we have $s = 2$. Recall that the roots of the characteristic equation are 3 and 2, each with multiplicity 1. Therefore $M = 1$, and the form of the particular solution is

$$a_n^{(p)} \;=\; n\, (d_2\, n^2 + d_1\, n + d_0)\, 2^n\,.$$

Now we can proceed as before.

Sections 9.1 and 9.5: Relations

This section also discusses the concept of *a relation from one set to another set.*

Examples:
(1) "speaks" is a relation *from the set of people to the set of languages* (we interpret this as "speaks fluently")
E.g. Justin Trudeau speaks English
Andrés Manuel López Obrador speaks Spanish
Justin Trudeau speaks French

The relation "speaks" can be viewed as a set of ordered pairs:

{ (Trudeau, English), (López Obrador, Spanish),
(Trudeau, French), (O Scholz, German), (R Sunak, English),
(R Sunak, Punjabi), ...}
$\subseteq$ { people } $\times$ { languages }

(2) "is greater than" is a relation from $\mathbb{R}$ to $\mathbb{R}$. E.g. $\sqrt{5} > -1.7$;
$19 > \pi$; $-4 \not> 2.311$ (i.e. $\neg -4 > 2.311$)

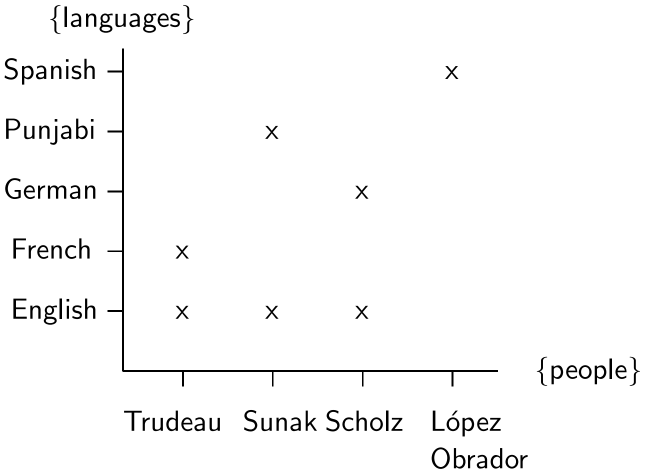
Definition: (Section 9.1.1) *Let A and B be sets. A relation R from A to B is a subset of $A \times B$.*

We write xRy to mean that x is related by R to y, i.e. that $(x, y) \in R$.

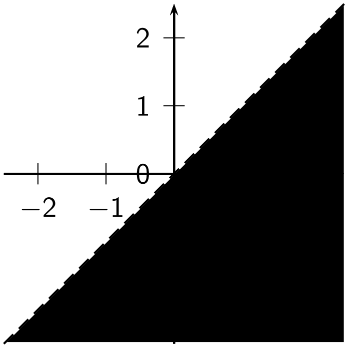
When A and B are the same set, we often say that "R is a relation on A" rather than "R is a relation from A to A." (E.g., "is greater than" is a relation on $\mathbb{R}$.) In this case, R is a subset of $A \times A$.

Usually, we think of a relation in terms of some kind of rule or condition that you can check for a given x and y (e.g., "is greater than", "speaks").
E.g., for "speaks": Is (Scholz, English) on the list of all pairs that define this relation? Ask Google; . . . apparently, yes.
[So is (Sunak, Hindi), but not (López Obrador, English).]

Since a relation R from A to B is a subset of $A \times B$, we can think it visually as a *graph*.
For example, for the relation "speaks" from the set of people to the set of languages:



Another example: For the relation “greater than“ from $\mathbb{R}$ to $\mathbb{R}$: $(x > y)$



Now we'll just consider relations on a set (i.e., from a set to itself.)

Examples: (3) "divides" on $\mathbb{Z}$: m divides n if $m \neq 0$ and $\frac{n}{m} \in \mathbb{Z}$.

(4) "speaks a language in common with": write the relation as LC:
Trrudea LC Scholz, Trudeau LC Sunak,
but $\neg$ (Trudeau LC López Obrador) .

(5) $2x^3 + y^2 = 3xy$. This equation defines a relation E on $\mathbb{R}$ by saying that xEy if and only if x and y satisfy this equation. That is, thinking of a relation on $\mathbb{R}$ as a set of ordered pairs of real numbers, we have

$$E = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, 2x^3 + y^2 = 3xy\}.$$

The graph of this relation is a curve in the xy-plane.

E.g. Is it true that $1E2$, i.e. that $(1, 2) \in E$? Check the equation: $2(1^3) + (2^2) = 6$ and $3(1)(2) = 6$. That is, $(1, 2)$ is a point on this curve. Therefore $1E2$.
Some other pairs in relation E: (0,0), (1,1), (-2,2), (-2,-8), $(\frac{5}{8}, \frac{5}{16})$.

A function is a special kind of relation. (Ses Subsection 9.1.2)

Let R be a relation from the set A to the set B.
When does R correspond to a function from A to B?

Answer: This happens if and only if the following is true:
For every x in A, there exists a unique y in B such that xRy (i.e., such that $(x, y) \in R$).

Here are three properties that a relation on a set could have:

- Reflexive
- Symmetric
- Transitive

Definition: The relation R on A is *symmetric* if

$$\forall x \, \forall y \, (xRy \; \rightarrow \; yRx) \, .$$

Which of the following relations are symmetric?
"Less than": $<$ on $\mathbb{R}$: No
"Equals": $=$ (on any set): Yes

Poll: Which if these is (are) NOT symmetric?

(A) "Not equal": $\neq$ (on any set)
(B) "Speaks a common language with"
(C) "Has the same length" on a set of strings Σ^*
(D) "divides": $x|y$ on $\mathbb{Z}^+$
(E) "Born in same country as" on the set of people

Answer: ????

Remark: If R is symmetric, then $\forall x \, \forall y \, (xRy \leftrightarrow yRx)$.

Next class: Read the rest of Section 9.1 and 9.5.

Homework updates:

- Problem Set C is due Thursday November 16.
- Homework assignment 6 (in Connect) is due Sunday March 12.