EECS/MATH 1019
Section 8.2: Solving Linear Recurrence Relations

March 7, 2023

Section 8.2 gives some recipes for finding solutions to some special kinds of recurrence relations, namely linear recurrence relations with constant coefficients. Some examples are

$$f_n \;=\; f_{n-1} + f_{n-2}$$

$$\text{and} \qquad g_n \;=\; 6\,g_{n-1} - 7\,g_{n-2} + 3.2\,g_{n-4}$$

$$\text{and} \qquad h_n \;=\; 1.6\,h_{n-1} + \,h_{n-2} - 13\,h_{n-3} \,+\, n^2 - 5\,.$$

These are all of the form

$$a_n \;=\; c_1\,a_{n-1} \,+\, c_2\,a_{n-2} \,+\, \cdots \,+\, c_k\,a_{n-k} \,+\, F(n)$$

where F is any function, the "coefficients" $c_1, \ldots, c_k$ are real constants (i.e., cannot depend on n), and k is the "degree" (assuming $c_k \neq 0$; note that the other c_i's could be 0, such as c_3 in the recursion for g_n).

The relation is homogeneous if $F(n) = 0$ for every n. (As for $\{f_n\}$ and $\{g_n\}$ above.)

8.2.1: Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

That is, we focus on relations of the form

$$a_n \;=\; c_1\, a_{n-1} \;+\; c_2\, a_{n-2} \;+\; \cdots \;+\; c_k\, a_{n-k}\,. \qquad (*)$$

<u>Observation 1:</u> Let r be a real number (constant). Then the sequence defined by $a_n = r^n$ satisfies $(*)$ if and only if

$$\begin{array}{rrcl} & r^n & = & c_1\, r^{n-1} + c_2\, r^{n-2} + \cdots + c_k\, r^{n-k} \quad \text{Divide by } r^{n-k}\text{:} \\ \leftrightarrow & \dfrac{r^n}{r^{n-k}} & = & c_1\, \dfrac{r^{n-1}}{r^{n-k}} + c_2\, \dfrac{r^{n-2}}{r^{n-k}} + \cdots + c_k\, \dfrac{r^{n-k}}{r^{n-k}} \\ & r^k & = & c_1\, r^{k-1} + c_2\, r^{k-2} + \cdots + c_k\, r^0 \end{array}$$

that is, if and only if r satisfies the equation

$$r^k \;-\; c_1\, r^{k-1} \;-\; c_2\, r^{k-2} \;-\; \cdots \;-\; c_{k-1} r \;-\; c_k \;=\; 0 \qquad (CE)$$

called the characteristic equation (CE) of the recurrence relation $(*)$. The solutions of (CE) are the characteristic roots.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}\,. \qquad (*)$$

Observation 2.1: If the sequence u_n is a solution of $(*)$, and b is a real number, then the sequence $b\,u_n$ is also a solution of $(*)$.

Observation 2.2: If the sequences u_n and v_n are two solutions of $(*)$, then the sequence $u_n + v_n$ is also a solution of $(*)$.

Observation 2.3: **Corollary:** If the sequences s_n and t_n are two solutions of $(*)$, and b and d are real numbers, then $b\,s_n + d\,t_n$ is also a solution of $(*)$.

See p. 541 for justification of the above.

We now look at the case $k = 2$ of $(*)$, i.e. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, and its associated characteristic equation $r^2 - c_1 r - c_2 = 0$.

Theorem 1: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1 r - c_2 = 0$ has two different roots r_1 and r_2. Then the solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are precisely the sequences $a_n = b\,r_1^n + d\,r_2^n$, where b and d are constants.

(See p. 542 for the proof.)

Theorem 1: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1 r - c_2 = 0$ has two different roots r_1 and r_2. Then the solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are precisely the sequences $a_n = b\, r_1^n + d\, r_2^n$, where b and d are constants.

Example D: Find the solution of

$$a_0 = 2, \quad a_1 = 9, \quad \text{and} \quad a_n = 5\, a_{n-1} - 6\, a_{n-2} \quad \text{for } n \geq 2.$$

Apply Theorem 1. The characteristic equation is $r^2 - 5r + 6 = 0$, Factoring this as $(r-3)(r-2) = 0$, we see that its roots are $r_1 = 3$ and $r_2 = 2$.
So the solution must be $a_n = b\, 3^n + d\, 2^n$ for some b and d.
Now what? Use the initial conditions to find b and d.

$$2 = a_0 = b\, 3^0 + d\, 2^0 \qquad \text{and} \qquad 9 = a_1 = b\, 3^1 + d\, 2^1 .$$

$$\text{That is, solve} \qquad 2 = b + d \quad \text{and} \quad 9 = 3b + 2d.$$

We can solve this and obtain $b = 5$ and $d = -3$. (Always check!)
So the solution is $a_n = 5 \cdot 3^n - 3 \cdot 2^n$.

See also Example 4 on pages 543–544, which solve the Fibonacci recurrence relation

$$f_0 = 0, \quad f_1 = 1, \quad \text{and} \quad f_n \;=\; f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

The characteristic equation is $r^2 - r - 1 = 0$, with characteristic roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

The resulting formula is

$$f_n \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Remarks: Some things to think about:

1. Is it surprising that this formula for f_n always produces an integer value?

2. If you had to evaluate f_{100}, would you use this exact formula or would you apply the recurrence relation 99 times?

We have been using

Theorem 1: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1\, r - c_2 \;=\; 0$ has two different roots r_1 and r_2. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ are precisely the sequences $a_n = b\, r_1^n + d\, r_2^n$, where b and d are constants.

What if there is only one root?

Theorem 2: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1\, r - c_2 \;=\; 0$ has only one real root r_0. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ are precisely the sequences $a_n = (b\, n + d)\, r_0^n$, where b and d are constants.

Theorem 2: Let $c_1, c_2 \in \mathbb{R}$. Assume that the equation $r^2 - c_1 r - c_2 = 0$ has only one real root r_0. Then the solutions of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ are precisely the sequences $a_n = (b n + d) r_0^n$, where b and d are constants.

Example E: Find the solution of

$$a_0 = 2, \quad a_1 = 10, \quad \text{and} \quad a_n = 4 a_{n-1} - 4 a_{n-2} \quad \text{for } n \geq 2.$$

The characteristic equation is $r^2 - 4r + 4 = 0$. Factoring this as $(r-2)^2 = 0$, we see that it has only one root, $r_0 = 2$.
So the solution must be $a_n = (bn + d) 2^n$ for some b and d.
Now use the initial conditions to find b and d.

$$2 = a_0 = (b(0) + d) 2^0 \qquad \text{and} \qquad 10 = a_1 = (b(1) + d) 2^1 .$$

$$\text{That is, solve} \qquad 2 = d \quad \text{and} \quad 10 = 2b + 2d.$$

We can solve this and obtain $d = 2$ and $b = 3$.
So the solution is $a_n = (3n + 2) 2^n$.

When the degree is not necessarily 2, we have two theorems that generalize Theorems 1 and 2.

Theorem 3: Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Assume that the characteristic equation $r^k - c_1\, r^{k-1} - c_2\, r^{k-2} - \ldots - c_{k-1}\, r - c_k = 0$ has k different roots $r_1, r_2, \ldots, r_k$. Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots + c_k\, a_{n-k}$ are precisely the sequences $a_n = b_1\, r_1^n + \ldots + b_k\, r_k^n$, where $b_1, \ldots, b_k$ are constants.

Example F: Find the solution of

$a_0=2,\ a_1=14,\ a_2=14$ and $a_n = -a_{n-1} + 4\, a_{n-2} + 4\, a_{n-3}\ \ (n \geq 3)$.

Apply Theorem 3. The characteristic equation is $r^3 + r^2 - 4r - 4 = 0$. Factoring this as $(r+1)(r^2-4) = 0$, we see that it has three roots: 2, -2, and -1.

Solution: $a_n = b_1\, 2^n + b_2(-2)^n + b_3(-1)^n$ for some b_1, b_2, b_3.
Now use the initial conditions to find b_1, b_2, and b_3.

$$\begin{aligned}
2 &= a_0 = b_1\, 2^0 + b_2(-2)^0 + b_3(-1)^0 = b_1 + b_2 + b_3 \\
14 &= a_1 = b_1\, 2^1 + b_2(-2)^1 + b_3(-1)^1 = 2b_1 - 2b_2 - b_3 \\
14 &= a_2 = b_1\, 2^2 + b_2(-2)^2 + b_3(-1)^2 = 4b_1 + 4b_2 + b_3\,.
\end{aligned}$$

Solve this system of equations:

$$\left.\begin{array}{rcr} 2 & = & b_1 + b_2 + b_3 \\ 14 & = & 2b_1 - 2b_2 - b_3 \\ 14 & = & 4b_1 + 4b_2 + b_3 \end{array}\right\} \overset{\text{solution:}}{\Longrightarrow} \begin{array}{rcr} b_1 & = & 5 \\ b_2 & = & -1 \\ b_3 & = & -2 \end{array}$$

Plugging this back into $a_n = b_1\, 2^n + b_2(-2)^n + b_3(-1)^n$, we find that the solution to

$$a_0{=}2,\ a_1{=}14,\ a_2{=}14 \text{ and } a_n = -a_{n-1} + 4\,a_{n-2} + 4\,a_{n-3} \quad (n \geq 3).$$

is $$a_n = 5(2^n) - (-2)^n - 2(-1)^n \quad (n = 0, 1, 2, \ldots).$$

The fourth theorem deals with the general situation of multiple roots.
For example, if the characteristic equation (CE) consists of polynomial of degree 6 and factors to

$$(r-2)^3(r-3.2)^2(r+4) \;=\; 0\,,$$

then the characteristic roots are 2, 2, 2, 3.2, 3.2, and -4.
We say that there are 6 real roots but only 3 distinct real roots.
We can label them $r_1 = 2$, $r_2 = 3.2$, and $r_3 = -4$.
(The order doesn't matter, as long as we declare one particular order and then always use it.)

For each i, we write m_i to denote the multiplicity of the root r_i, which is the number of times that r_i occurs as a root in the characteristic equation.
In this example, $m_1 = 3$, $m_2 = 2$, and $m_3 = 1$.
Observe that $m_1 + m_2 + m_3$ must equal the number of roots of the characteristic equation, which is 6 in this example.

Theorem 4: Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Assume that the characteristic equation $r^k - c_1\, r^{k-1} - c_2\, r^{k-2} - \ldots - c_{k-1}\, r - c_k \;=\; 0$ has k real roots, but only t distinct roots $r_1, r_2, \ldots, r_t$. For $i = 1, \ldots t$, let m_i be the multiplicity of the root r_i.
(*Note:* $\sum_{i=1}^{t} m_i = k$ *because there are* k *real roots.*)
Then the solutions of $a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots + c_k\, a_{n-k}$ are precisely the sequences $a_n = p_i(n)\, r_1^n + \ldots + p_t(n)\, r_t^n$, where p_i is a polynomial of degree at most $m_i - 1$.

For our previous example in which the characteristic equation factors to $(r-2)^3(r-3.2)^2(r+4) \;=\; 0$, the solution of the recurrence relation must have the form

$$\begin{aligned} a_n \;&=\; (b_{1,0} + b_{1,1}\, n + b_{1,2}\, n^2)\,(2^n) \\ &\quad + \; (b_{2,0} + b_{2,1}\, n)\,(3.2)^n \quad + \; b_{3,0}(-4)^n \end{aligned}$$

where each $b_{i,j}$ is a real constant. The values of the $b_{i,j}$'s are determined by the six equations of the initial conditions (typically values of a_0, a_1, a_2, a_3, a_4, and a_5, since this is a recurrence relation of degree 6). See text for examples.

8.2.3: Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Now we focus on relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (*_F)$$

where the function F is not identically 0.

E.g. $\quad (i) \quad s_n = 1.05\, s_{n-1} - 70 \qquad$ (recall Example C)

$\quad (ii) \quad a_n = 5\, a_{n-1} - 6\, a_{n-2} + n^2$

Removing the $F(n)$ term from $(*_F)$ produces the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (*_H)$$

E.g. for (i), we get $\quad s_n = 1.05\, s_{n-1} \qquad (*_H)$

For (ii), we get $\quad a_n = 5\, a_{n-1} - 6\, a_{n-2} \qquad (*_H)$.

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_k\, a_{n-k} + F(n) \qquad (*_F)$$
$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_k\, a_{n-k} \qquad (*_H)$$

Theorem 5: Let $\{a_n^{(p)}\}$ be any one particular solution of $(*_F)$. Then *every* solution of $(*_F)$ is of the form $a_n^{(p)} + a_n^{(h)}$ where $a_n^{(h)}$ is a solution of $(*_H)$.

E.g. In Example (C), we found the the value s_n of Sandra's investments after n years (in thousands of dollars) satisfies

$$s_0 = 500 \quad \text{and} \quad s_n = 1.05\, s_{n-1} - 70 \quad (*_F) \quad \text{for } n \geq 1.$$

The associated homogeneous recurrence $(*_H)$ is $s_n = 1.05\, s_{n-1}$.
Poll: The most general form of the solution of $(*_H)$ is
(A) $s_n = (1.05)^n$ (B) $s_n = b(1.05)^n$ for any real constant b
(C) $s_n = 1.05\, n$ (D) $s_n = b(1.05)^n + d$ for any real b and d

Answer: ?????

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (*_F)$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (*_H)$$

Theorem 5: Let $\{a_n^{(p)}\}$ be any one particular solution of $(*_F)$. Then *every* solution of $(*_F)$ is of the form $a_n^{(p)} + a_n^{(h)}$ where $a_n^{(h)}$ is a solution of $(*_H)$.

$$s_0 = 500 \quad \text{and} \quad s_n = 1.05\, s_{n-1} - 70 \quad (*_F) \quad \text{for } n \geq 1.$$

We need to find one particular solution of $(*_F)$. Idea/Guess: How much money would Sandra need for her investment total to stay constant each year? We have $s_0 = s_1 \quad \leftrightarrow \quad s_0 = 1.05\, s_0 - 70$ $\leftrightarrow \quad 70 = 0.05 s_0 \qquad \leftrightarrow \quad s_0 = 70/0.05 = 1400.$

So a particular solution of $(*_F)$ is $s_n^{(p)} = 1400$ for every n.
From the previous slide, $a_n^{(h)} = b(1.05)^n$ for some real b.
By Theorem 5, the solution of $(*_F)$ with $s_0 = 500$ must be of the form $s_n = b\,(1.05)^n + 1400.$
To find b, solve $500 = s_0 = b(1.05)^0 + 1400 = b + 1400.$
$\therefore b = 500 - 1400 = -900.$ Solution: $s_n = 1400 - 900(1.05)^n.$

Next class: We will finish Section 8.2 and begin Section 9.1.
(The only sections of Chapter 9 that will do are 9.1 and 9.5.)

Homework updates:

- Homework assignment 6 (in Connect) is due Sunday March 12.
- Problem Set C is due Thursday March 16.