EECS/MATH 1019
Section 5.3 (concluded): Structural Induction
Section 8.1: Recurrence Relations — Part I

March 2, 2023

Recap of **Recursively Defined Sets and Structures (sec. 5.3.3)**

Examples from previous class:

Example T: Consider the set T that is defined by the following rules:

(i) *Basis step:* $1 \in T$.

(ii) *Recursive step:* If $x \in T$, then $x + 2 \in T$.

(We always implicitly assume (iii): Our set has no other elements besides those that can be obtained through steps (i) and (ii).)

$1 \in T$, therefore $1 + 2 \in T$, i.e. $3 \in T$;
and therefore $3 + 2 \in T$, i.e. $5 \in T$;
and therefore $5 + 2 \in T$, i.e. $7 \in T$;
and so on ...

We can see that T is the set of odd positive integers.

Example V: Consider the set V that is defined by the following rules:

(i) *Basis step:* $1 \in V$.

(ii) *Recursive step:* If $x \in V$, then $x + 2 \in V$ and $x - 2 \in V$.

$1 \in V$; therefore taking $x = 1$ in step (ii), we see that
$1 + 2 \in V$ and $1 - 2 \in V$, i.e. $3 \in V$ and $-1 \in V$.

Now we can take x to be 3 or -1 in step (ii).
Taking $x = 3$ first, we see that
$3 + 2 \in V$ and $3 - 2 \in V$, i.e. $5 \in V$ and $1 \in V$;
taking $x = -1$ next, we see that
$-1 + 2 \in V$ and $-1 - 2 \in V$, i.e. $1 \in V$ and $-3 \in V$.

Now the new possibilities for x are 5 and -3.
So V contains $5 + 2$, $5 - 2$, $-3 + 2$, and $-3 - 2$.

Now the new possibilities for x are 7 and -5. And so on...

We see that V is the set of all odd integers (positive and negative).

Example W:

(*i*) *Basis step:* $3 \in W$.

(*ii*) *Recursive step:* If $x \in W$, then $x + 12 \in W$ and $x - 4 \in W$.

$3 \in W$; therefore taking $x = 3$ in step (ii), we see that
$3 + 12 \in W$ and $3 - 4 \in W$, i.e. $15 \in W$ and $-1 \in W$.

Now we can take x to be 15 or -1 in step (ii).
Taking $x = 15$ first, we see that
$15 + 12 \in W$ and $15 - 4 \in W$, i.e. $27 \in W$ and $11 \in W$;
taking $x = -1$ next, we see that
$-1 + 12 \in W$ and $-1 - 4 \in W$, i.e. $11 \in W$ and $-5 \in W$.

Now the new possibilities for x are 27, 11 and -5.
So W contains $27 + 12$, $27 - 4$, $11 + 12$, $11 - 4$, $-5 + 12$ and $-5 - 4$.

Now the new possibilities for x are 39, 23, 7, and -9. And so on...

It may not be so clear yet exactly what the set W is. For example, is 1 an element of W? We now introduce a method called *structural induction* that lets us prove some things about W.

Structural Induction (subsection 5.3.4)

To prove that a statement $P(x)$ is true for every element x of a recursively defined set:

(i') Show that $P(x)$ is true for each element x specified in the basis step; and
(ii') Show that if $P(x)$ is true for each element x used to define a new element y in the recursive step, then $P(y)$ is true.

Example W:

(i) *Basis step:* $3 \in W$.
(ii) *Recursive step:* If $x \in W$, then $x + 12 \in V$ and $x - 4 \in W$.

We have seen that the set W contains -9, -5, -1, 3, 7, 11, 15, 23, 27, and 39 (and infinitely many more elements!)
We shall prove that every element of W is 3 more than a multiple of 4. That is, every element of W is of the form $3 + 4m$ for some integer m.

Let $P(x)$ be the statement "$x = 3 + 4m$ for some integer m."
Then $P(3)$ is true $(3 = 3 + 4 \times 0)$, as is $P(7)$ $(7 = 3 + 4 \times 1)$, $P(39)$ $(39 = 3 + 4 \times 9)$, $P(-9)$ $(-9 = 3 + 4 \times (-3))$, etc.

(*i*) *Basis step:* $3 \in W$.
(*ii*) *Recursive step:* If $x \in W$, then $x + 12 \in W$ and $x - 4 \in W$.

Let $P(x)$ be the statement $x = 3 + 4m$ for some integer m.

The steps for structural induction:

(*i*') Show that $P(x)$ is true for each element x specified in the basis step; and
(*ii*') Show that if $P(x)$ is true for each element x used to define a new element y in the recursive step, then $P(y)$ is true.

For (*i*'): Show that $P(3)$ is true. DONE.

For (*ii*'): Assume that $x \in W$ and $P(x)$ is true, and show that $P(x + 12)$ and $P(x - 4)$ are true.

So assume $x \in W$ and $x = 3 + 4m$ for some integer m.
Then $x + 12 = (3 + 4m) + 12 = 3 + 4m + 4(3) = 3 + 4(m + 3)$.
And $m + 3$ is an integer, so $P(x + 12)$ is true.
Similarly, $x - 4 = (3+4m) - 4 = 3+4m + 4(-1) = 3 + 4(m - 1)$.
And $m - 1$ is an integer, so $P(x - 4)$ is true.

So step (*ii*') is DONE.
We conclude that $P(x)$ is true for every x in W.

Our next example involves **strings**. Recall from last class:

Let Σ be a set (usually finite). A string over Σ is a finite sequence of elements of Σ. (Order matters!)

E.g. if $\Sigma = \{a, b, c, d\}$, then some strings over Σ are *abdba*, and *baadb*, and *cccacaac*, and *d*, and *aaa*.

The number of symbols in a string is its length.

We call Σ the alphabet, and we call its elements symbols or characters.

The empty string is a string with no symbols; we denote it by λ. The length of λ is 0.

If s and t are strings over Σ, then their concatenation st is the string obtained by writing t immediately to the right of s.

E.g. For $\Sigma = \{a, b, c, d\}$: Let $s = dabbd$ and $t = cca$.
Then $st = dabbdcca$ and $ts = ccadabbd$.

For any string w, we have $w\lambda = \lambda w = w$.

The set of all strings over Σ is denoted Σ^*. It is defined recursively by

(i) *Basis step:* $\lambda \in \Sigma^*$.

(ii) *Recursive step:* If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$.

Example: For $\Sigma = \{a, b, c, d\}$, we see that Σ^* contains λa, λb, λc, and λd, which equal a, b, c, and d respectively.
Taking $w = a$, we see that Σ^* contains aa, ab, ac, and ad.
Taking $w = b$, we see that Σ^* contains ba, bb, bc, and bd.
Similarly, Σ^* contains ca, cb, cc, cd, da, db, dc, and dd.

Taking $w = aa$, we see that Σ^* contains aaa, aab, aac, and aad.
Taking $w = ab$, we see that Σ^* contains aba, abb, abc, and abd.
And so on...

Example: A binary string is a string over the alphabet $\{0, 1\}$.
Then $\{0, 1\}^*$ is the set of all binary strings.

Example: In the previous class, we saw this recursive definition of a particular subset E of bit strings.
(i) *Basis step:* $\lambda \in E$.
(ii) *Recursive step:* If $w \in E$, then $w1 \in E$, $1w \in E$, and $0w0 \in E$.

So E contains λ, as well as the following strings:
(Taking w to be λ:) $\lambda 1$, 1λ, and $0\lambda 0$; that is, 1 and 00;
(Taking w to be 1:) 11, 11, and 010; that is, 11 and 010;
and
(taking w to be 00:) 001, 100, and 0000; that is, 001, 100, and 0000;
and so on...

Question: Is 10111101110 in E?
We claim it is not, because every string in E must have an even number of 0's.
We shall prove this using Structural Induction.

(i) *Basis step:* $\lambda \in E$.
(ii) *Recursive step:* If $w \in E$, then $w1 \in E$, $1w \in E$, and $0w0 \in E$.

For a bit string x, let $P(x)$ be the statement x has an even number of 0's.

To use structural induction to prove that $P(x)$ is true for every x in E, we need to:

(i') Show that $P(x)$ is true for each x in the basis step; and
(ii') Show that if $P(x)$ is true for each x used to define a new element y in the recursive step, then $P(y)$ is true.

For (i'): Show that $P(\lambda)$ is true. The number of 0's in λ (the empty string) is 0, which is even. DONE.

For (ii'): Let $w \in E$, and assume that $P(w)$ is true. That is, let z be the number of zeroes in w, and assume that z is even. Then:
the number of 0's in $w1$ is z, which is even. $\therefore$ $P(w1)$ is true;
the number of 0's in $1w$ is z, which is even. $\therefore$ $P(1w)$ is true; and
the number of 0's in $0w0$ is $z+2$, which is even. $\therefore$ $P(0w0)$ is true.
And so step (ii') is DONE. This completes the structural induction. Therefore $P(x)$ is true for every x in E.

(i) *Basis step:* $\lambda \in E$.
(ii) *Recursive step:* If $w \in E$, then $w1 \in E$, $1w \in E$, and $0w0 \in E$.

We have proved that every string in E has an even number of 0's.

Converse question: Does every string with an even number of 0's have to be in E? We have not addressed this question yet.

(It turns out that the answer is yes. One way to prove it uses the well-ordering property, as follows.
If the answer is no, then let s be a shortest possible string with an even number of 0's that is **not** in E. Since $s \notin E$, the length of s must be at least 2. Any string of length 2 or more must have one of the following forms: $1w1$ or $1w0$ or $0w1$ or $0w0$.
<u>Exercise:</u> Finish this proof.)

Section 8.1: Recurrence relations

In this text, a *recurrence relation* is a recursively defined sequence.

Example A: What is the number of strings of length n over an alphabet Σ?

Recall: We can recursively define Σ^* (the set of strings over Σ) by
(i) *Basis step:* $\lambda \in \Sigma^*$.
(ii) *Recursive step:* If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$.

E.g. For $\Sigma = \{a, b, c, d\}$, the recursive step says that if $w \in \Sigma^*$, then wa, wb, wc, and wd are all in Σ^*.

For $n \in \mathbb{N}$, let s_n be the number of strings of length n in $\{a, b, c, d\}^*$.
Then $s_0 = 1$ (the empty string), and we see from the recursive step that $s_n = 4s_{n-1}$ for $n \geq 1$ (since every string w of length $n-1$ produces 4 strings of length n, which is $4s_{n-1}$ strings in total, and they are all different (if w and u are two different strings of length $n-1$, then wx cannot equal ux)).

There is only one sequence that satisfies $s_0 = 1$ and $s_n = 4s_{n-1}$ for all $n \geq 1$, and it is $s_n = 4^n$.

More generally, if the number of symbols in the alphabet Σ is L, and s_n is the number of strings of length n in Σ^*, then the sequence $\{s_n\}$ satisfies the recurrence relation then

$$s_0 = 1 \quad \text{and} \quad s_n = L\, s_{n-1} \quad \text{for } n \geq 1,$$

which tells us that $s_n = L^n$ for every $n \in \mathbb{N}$. That is, $s_n = L^n$ is the unique sequence that satisfies the above recurrence relation.

Example B: Let A be the set of all bit strings that do not have any adjacent 0's (i.e., no 0's next to each other).

The shortest strings in A are

Length 0:	λ
Length 1:	0, 1
Length 2:	01, 10, 11
Length 3:	010, 110, 101, 011, 111

Question: How many strings of length n are in A? (Ex. 3, p. 531.)

Let b_n be the number of bit strings of length n have no adjacent 0's. What can we say about b_n?
We know from above that $b_1 = 2$, $b_2 = 3$, $b_3 = 5$ and $b_0 = 1$.
What about larger values of n?

Let b_n be the number of bit strings of length n have no adjacent 0's. What can we say about b_n?
We know $b_1 = 2$ and $b_0 = 1$. What about $n \geq 2$?

Let $b_n^{(1)}$ be the number of bit strings of length n that have no adjacent 0's, and end in 1.
Let $b_n^{(0)}$ be the number of bit strings of length n that have no adjacent 0's, and end in 0.

Then $b_n = b_n^{(1)} + b_n^{(0)}$ for every n.

Every bit string of length n with no adjacent 0's and ending in 1 is of the form $w1$ where w has length $n-1$ and has no adjacent 0's and has no other restrictions. Thus there is a one-to-one correspondence between strings counted by $b_n^{(1)}$ and strings counted by b_{n-1}. Therefore $b_n^{(1)} = b_{n-1}$.
Every bit string of length n with no adjacent 0's and ending in 0 is of the form $u10$ where u has length $n-2$ and has no adjacent 0's and has no other restrictions. Therefore $b_n^{(0)} = b_{n-2}$.

Therefore $\quad b_n = b_n^{(1)} + b_n^{(0)} = b_{n-1} + b_{n-2} \qquad$ (for $n \geq 2$).

We have shown

$$b_0 = 1, \quad b_1 = 2, \quad \text{and} \quad b_n = b_{n-1} + b_{n-2} \quad \text{for all } n \geq 2.$$

This is the same recurrence relation for the Fibonacci numbers $\{f_n\}$, except they had different initial values: $f_0 = 0$ and $f_1 = 1$, which gave $f_2 = 1$, $f_3 = 2$, $f_4 = 3, \ldots$

So we can relate the Fibonacci sequence to the sequence $\{b_n\}$ by $b_n = f_{n+2}$.

So we now have an answer to our question:

The number of bit strings of length n with no adjacent 0's is the Fibonacci number f_{n+2}.

Example C: Sandra is planning to retire soon. She thinks she will have \$500,000 in investments at the time of her retirement.
She expects the investments to grow by 5% each year, and she plans to withdraw \$70,000 from the investments at the end of each year.
How much money will she have n years after she retires?

Poll: Let x_n be the amount of money after n years of retirement (in thousands of dollars).
Then $x_0 = 500$, and for each $n \geq 1$ we have

(A) $x_n \;=\; 0.05\, x_{n-1} - 70$

(B) $x_n \;=\; 0.05\, x_{n-1} - 0.05(70)$

(C) $x_n \;=\; x_{n-1} - 1.05(70)$

(D) $x_n \;=\; 1.05\, x_{n-1} - 70$

Answer: (D). E.g. After 1 year, Sandra has
$x_1 = 1.05\, x_0 - 70 = 1.05(500) - 70 = 525 - 70 = 455.$
After 2 years, Sandra has
$x_2 = 1.05\, x_1 - 70 = 1.05(455) - 70 = 477.75 - 70 = 407.75.$

See Subsection 8.1.2 for more examples of how recurrence relations can arise. You can omit Subsection 8.1.3, which is about algorithms (although you should read it if you are interested!)

Next class: Read Section 8.2.

Homework updates:

- Homework assignment 5 (in Connect) is due Sunday March 5.
- Problem Set B is due Sunday March 5.