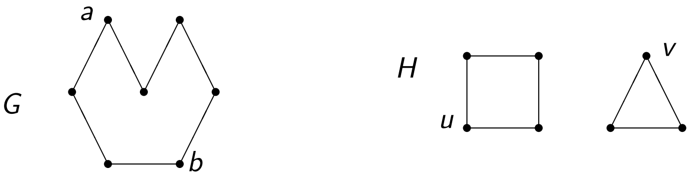
EECS/MATH 1019
Section 10.4:
Paths and Connectivity

March 30, 2023

As previewed in Tuesday's class: Here are two graphs, both of which have 7 vertices, all of degree 2, and 7 edges:

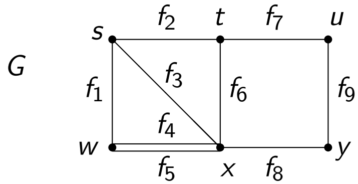


These two graphs seem pretty different. It is not hard to show that G and H are not isomorphic.

We shall introduce some definitions that reflect a major qualitative difference between G and H, namely that G is "connected" while H is "disconnected." Also, the graph H has two "connected components," with vertex u in one component and v in the other.

We say that G contains a "path" from a to b, but H does not contain a path from u to v.

Now we shall give the proper definitions.

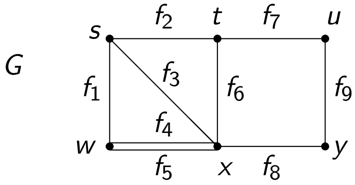


Definitions: Let G be an undirected graph.
Let $n \in \mathbb{N}$, and let b and c be vertices of G.
A path in G of length n from b to c is a sequence $e_1, \ldots, e_n$ of n edges of E for which there is a sequence of vertices $z_0, z_1, \ldots, z_n$ such that $z_0 = b$, $z_n = c$, and z_{i-1} and z_i are the endpoints of e_i for each $i = 1, \ldots, n$.

In the above example, there are many paths from s to y, including
(1) f_2, f_6, f_8 (corresponding to vertices s, t, x, y);
(2) f_3, f_4, f_5, f_8 (corresponding to vertices s, x, w, x, y);
(3) f_1, f_4, f_6, f_6, f_8 (corresponding to vertices s, w, x, t, x, y).
(4) f_3, f_6, f_2, f_3, f_8 (corresponding to vertices s, x, t, s, x, y).

Definition (continued): A path is simple if no edge occurs more than once. (E.g. (1) and (2) are simple, but (3) and (4) are not simple.)

What is a path of length 0? It must go from a vertex to itself, with no edges.



A circuit (or cycle) is a path of non-zero length from a vertex to itself (i.e., having $b = c$).

E.g. f_7, f_9, f_8, f_3, f_2 is a circuit (from t to t).

If there are no multiple edges involved, we can just use the vertex sequence to describe a path or circuit. E.g. the above circuit is given by t, u, y, x, s, t.

Theorem 10.A: Let G be an undirected graph. Assume there is a path from b to c in G (where b and c are vertices of G). Then there is a simple path from b to c.

(Recall: A path is simple if no edge occurs more than once.)

Proof: Consider a path with the shortest length among all paths from b to c. Let the sequence of vertices in this path be $x_0, x_1, \ldots, x_n$. We claim that this (shortest) path is simple.

Assume that this shortest path is not simple.
Then it uses some edge twice, and hence it uses some vertex twice, say $x_i = x_j$ where $i < j$.

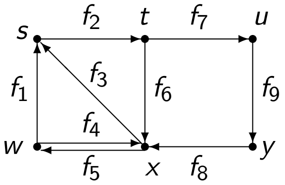
The part of the path from x_i to x_j is a circuit with $j - i$ edges, so we will remove it from the path, leaving the vertex sequence $x_0, x_1, \ldots, x_{i-1}, x_i(= x_j), x_{j+1}, \ldots, x_n$. This is a path of length $n - (j - i)$ from b to c, which is shorter that the path we chose.
This is a contradiction.
Therefore the (shortest) path we chose must have been simple.

Q.E.D.

We have similar terminology for directed graphs as well. We need to pay attention to the direction of each edge.

For the sequence of edges $e_1, \ldots, e_n$ to be a path in a digraph, the associated sequence of vertices $x_0, \ldots, x_n$ must be such that $e_k = (x_{k-1}, x_k)$ for each k (that is, e_k starts at x_{k-1} and ends at x_k).

D

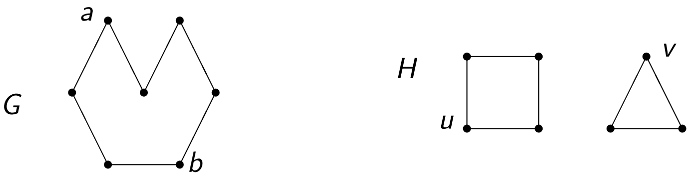


E.g. f_6, f_5, f_4 is a simple path in D, but f_6, f_8, f_9 is not a path in D. And f_2, f_7, f_9, f_8, f_3 is a circuit (with vertices s, t, u, y, x, s), but f_7, f_9, f_8, f_6 is not a circuit (edge f_6 goes the wrong way).

Example: In the directed graph representing the internet, in which each directed edge represents a hyperlink from one web page to another, a path corresponds to a sequence of clicks you can make, taking you through a sequence of pages, without using the back button.

Connectedness in Undirected Graphs

Definition: Let G be an *undirected* graph. We say that G is connected if there are paths from every vertex to every other vertex. We say that G is disconnected if it is not connected.



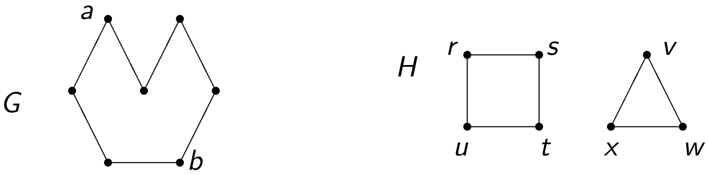
E.g. G is connected, and H is disconnected.

Recall that Theorem 10.A said that if there is a path from b to c in the undirected graph G, then there is a simple path from b to c. An immediate corollary is the following.

Theorem 1 (p. 717): In a connected undirected graph, there are *simple* paths from every vertex to every other vertex.

Consider an undirected graph $G = (V, E)$.
We can define a **relation** C on V by

$u\, C\, v$ if and only if there is a path from u to v.



E.g. For vertex r of H, we have rCs, rCt, and rCu, but $r \not C v$, $r \not C w$, and $r \not C x$.

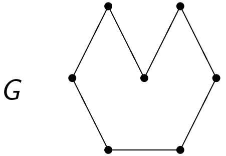
What properties does C have, in a general undirected graph?
Reflexive? (vCv for all v) Yes, since paths can have length 0.

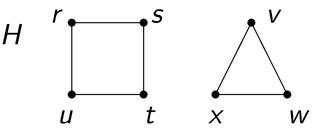
Symmetric? ($uCv \rightarrow vCu$) Yes, since edges are not directed.

Transitive? ($uCv \wedge vCw \rightarrow uCw$) Yes, since a path from u to v can be joined to a path from v to w to make a path from u to w.

Therefore C is an equivalence relation.

For an undirected graph $G = (V, E)$, we define a **relation** C on V by $u\ C\ v$ if and only if there is a path from u to v.





Poll: Which of the following sets is an equivalence class of C in the graph H?

(A) $\{r, s\}$ (B) $\{x, v, w\}$

(C) $\{u, t, x, w\}$ (D) $\{r, s, t, u, v, w, x\}$ Answer: (B).

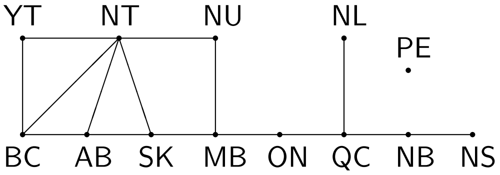
For a general undirected graph G, each equivalence class of this relation C is a set of vertices all of which are joined by paths to each other but not to any other vertices.

In particular, each equivalence class induces a connected subgraph, and it is not contained in any larger connected subgraph. (The text refers to such a subgraph as a maximal connected subgraph of G.)

What are the equivalence classes of the example G above? There is just one equivalence class, namely V.

The subgraphs induced by equivalence classes of the relation "there is a path from u to v" are called the connected components of the graph.

E.g. What are the connected components in the Canada graph?



There are two connected components: $\{PE\}$, and the subgraph obtained by removing PE.

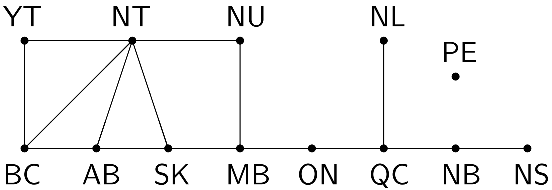
Definition: A cut edge of an undirected graph G is an edge whose removal produces a graph with more components than G. (Remember: when we remove an edge, we keep all the vertices!)

E.g. {MB,ON} is a cut edge, but {MB,NU} is not.

Poll: How many cut edges does the Canada graph have?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

<u>Answer:</u> (D). Note that {QC,NL} is a cut edge, because the vertex NL remains after the edge is removed. Similarly for {NB,NS}.

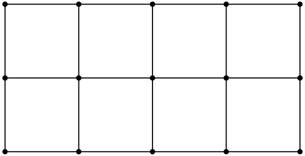


Definition: A cut vertex of an undirected graph G is a vertex whose removal produces a graph with more components than G. (Remember: when we remove a vertex, we must also remove all edges incident with that vertex!)
E.g. NB is a cut vertex, but NS is not.

Poll: How many cut vertices does the Canada graph have?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

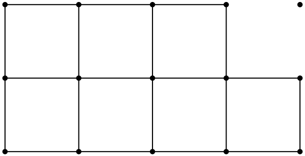
Answer: (C). They are NB, QC, ON, and MB.

Cut edges and cut vertices are relevant for assessing reliability of networks.

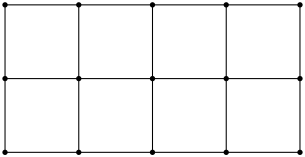


If our graph represents a communications network, or a road network connecting towns, then the loss of one vertex or edge (communications down, or road washed out) might be compensated for by routing around the lost part.

The above graph remains connected if any one edge is removed. What if two edges are removed? Removing two edges could disconnect the graph; for example,



Thus the minimum number of edges needed to disconnect this graph is 2.



We define the edge connectivity of an undirected graph G to be the smallest number of edges needed to disconnect the graph by their removal. We denote it $\lambda(G)$.

Poll: Suppose a graph J has a vertex of degree 5. What can you say about $\lambda(J)$?

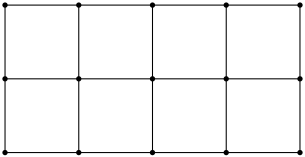
(A) $\lambda(J) = 5$ (B) $\lambda(J) \leq 5$ (C) $\lambda(J) \geq 5$

(D) This information tells us nothing about $\lambda(J)$.

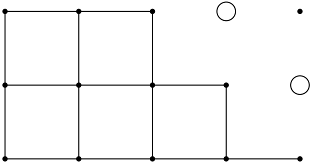
Answer: (B) (since we can disconnect J by removing the 5 edges incident on this vertex)

"**Corollary:**" $\lambda(J)$ is less than or equal to the smallest degree in J.

Now we consider removing *vertices*.



The above graph remains connected if any one vertex is removed. What if two vertices are removed? Removing two vertices could disconnect the graph; for example,

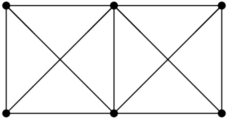


Thus the vertex connectivity of this graph equals 2.
We define the vertex connectivity of an undirected graph G to be the smallest number of vertices needed to disconnect the graph by their removal. We denote it $\kappa(G)$.

Theorem 10.B: For any undirected graph G, $\kappa(G) \leq \lambda(G)$.

$\kappa(G)$ = vertex connectivity, $\lambda(G)$ = edge connectivity

In the previous example, $\kappa(G) = \lambda(G) = 2$. Can we find an example in which $\kappa(G) \neq \lambda(G)$? By Theorem 10.B, this could only happen if $\kappa(G) < \lambda(G)$, i.e. the number of vertices you need to remove to disconnect G is less than the number of edges you would need to remove.

G 

Poll: What holds for this graph?
(A) $\kappa(G) = 2$ and $\lambda(G) = 2$
(B) $\kappa(G) = 2$ and $\lambda(G) = 3$
(C) $\kappa(G) = 3$ and $\lambda(G) = 3$
(D) $\kappa(G) = 3$ and $\lambda(G) = 4$

Answer: (B).

Exercise: Prove that (B) is true.

Next class: Read Section 10.5.

The current Connect assignment is due Sunday April 2.

There will be one more Connect assignment after this one, which will be due April 10.

Please fill out the course evaluations for each of your courses:
http://courseevaluations.yorku.ca/
The course evaluation questions are listed under two separate tabs. The first tab contains the Core Institutional Questions and the second tab has the Course Level Questions.
Please complete both sections of the evaluation form.