# EECS/MATH 1019
# Section 10.2: Introduction to Graphs – part 2

March 21, 2023

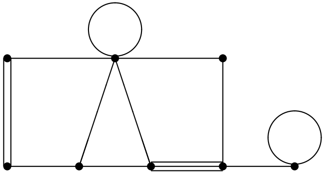
Recap: A graph is determined by a set of vertices and a set of edges.

Each edge connects two vertices, which are the endpoints of the edge.

We often write $V$ to denote the set of vertices and $E$ to denote the set of edges. Then the graph $G$ is written as $G = (V, E)$.

In general, a graph can have multiple edges between a pair of vertices. Such a graph is called a multigraph.

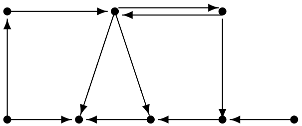
A graph can also have "loops." A loop is an edge whose two endpoints are the same vertex. For example, this graph has two loops and two sets of multiple edges:



A simple graph is a graph with no loops and no multiple edges.

Recall Example 3: The Internet  The vertices are web pages.

We represent a hyperlink from one page that points to another page by an edge. But the direction is important. We need directed edges (or arcs) to represent hyperlinks.



A directed graph or digraph $(V, E)$ consists of a set of vertices $V$ and a set of directed edges $E$.
Each directed edge is associated with an ordered pair of vertices $(u, v)$, and we say that this edge starts at $u$ and ends at $v$.

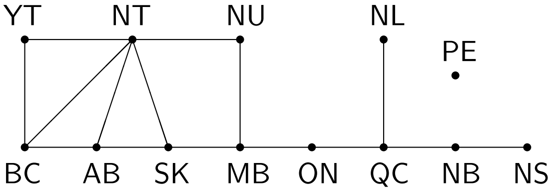
We can have an edge from $u$ to $v$ and another edge from $v$ to $u$ (as in above diagram).
We can also have multiple directed edges (directed multigraph), and some edges directed and others undirected (mixed graph), but these kinds of graph are used less often.

## More terminology for graphs (subsection 10.2.2)

For <u>undirected</u> graphs:

Suppose an edge $e$ has endpoints $u$ and $v$. Then we say that $e$ is incident with $u$ and with $v$, that $u$ and $v$ are adjacent, and that $u$ and $v$ are neighbours.

The set of all neighbours of a vertex $v$ is called the neighbourhood of $v$, and is denoted $N(v)$.

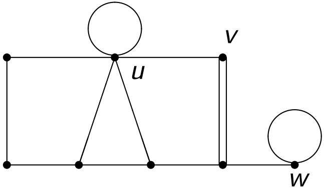



E.g. $N(QC) = \{$ ON, NL, NB $\}$, and $N(PE) = \emptyset$.

If $A \subseteq V$, then $N(A)$ is the set of all the neighbours of the vertices of $A$. That is, $N(A) = \bigcup_{v \in A} N(v)$.

E.g. $N(\{$BC, AB, NS$\}) = \{$YT, NT, SK, BC, AB, NB$\}$.

In an undirected graph, the degree of a vertex $v$ is the number of edges incident with $v$, but we count each loop twice. We write it $\deg(v)$.




In this graph, $\deg(w) = 3$, $\deg(u) = 6$, and $\deg(v) = 3$.

One reason that we treat loops like this is because it allows us to obtain the following neat fact:

**Theorem 1**: Let $G$ be an undirected graph with $m$ edges. Then the sum of the degrees of all of the vertices is equal to $2m$.

E.g. in the above graph, $m = |E| = 13$, and the sum of the degrees is $2 + 6 + 3 + 2 + 3 + 3 + 4 + 3 = 26$.

Idea of proof: count the number of . ("half-edges")

**Theorem 1**: Let $G$ be an undirected graph with $m$ edges. Then the sum of the degrees of all of the vertices is equal to $2m$.

**Poll:** Does there exist an undirected graph with 7 vertices, each of which has degree 3?
(A) Yes
(B) No

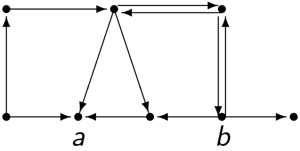
Answer: NO. If such a graph existed, then twice the number of edges would be $3+3+3+3+3+3+3 = 21$, which is not twice an integer.

More generally, we have

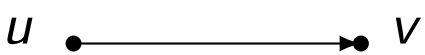
**Theorem 2**: Let $G$ be an undirected graph. Then the number of vertices of odd degree must be even.

(This is because the sum of all the degrees must be an even number.)

We also need analogous definitions for directed graphs.




Let $G = (V, E)$ be a directed graph.

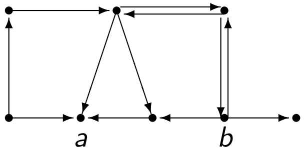



Consider an edge in $G$ starting at vertex $u$ and ending at vertex $v$.
Recall that we associate this edge with the ordered pair $(u, v)$.
We call $u$ the initial vertex and $v$ the terminal vertex (or end vertex) of the edge.
We also say that *$u$ is adjacent to $v$* and *$v$ is adjacent from $u$*.

The out-degree of a vertex $w$ is the number of edges whose initial vertex is $w$. The in-degree of a vertex $w$ is the number of edges whose terminal vertex is $w$.

We write $\deg^+(w)$ for the out-degree of $w$, and $\deg^-(w)$ for the in-degree of $w$.

E.g. in the above graph, we have $\deg^-(a) = 3$, $\deg^+(a) = 0$, $\deg^-(b) = 1$, and $\deg^+(b) = 3$.




Let $G = (V, E)$ be a directed graph.

$\deg^+(w)$ is the out-degree and $\deg^-(w)$ is the in-degree of $w$

Suppose $\sum_{w \in V} \deg^+(w) = 100$.
What can we deduce about the digraph $G$?

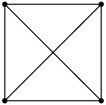
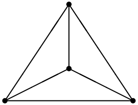
Answer: It tells us that $\sum_{w \in V} \deg^-(w) = 100$ and the set of (directed) edges $E$ contains exactly 100 edges. This is because each edge $(u, v)$ contributes 1 to the out-degree of $u$ and 1 to the in-degree of $v$. More generally:

**Theorem 3** Let $G = (V, E)$ be a digraph. Then

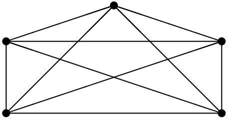
$$\sum_{w \in V} \deg^-(w) = \sum_{w \in V} \deg^+(w) = |E|.$$

## 10.2.3: Some special graphs

For $n \in \mathbb{Z}^+$, the complete graph on $n$ vertices has $n$ vertices and one edge between each pair of distinct vertices. This graph is denoted $K_n$.

$K_4$  or 

Remarks: (1) There are many ways to draw any one graph.
(2) Edges can cross without there being a vertex at the crossing.

$K_5$ 

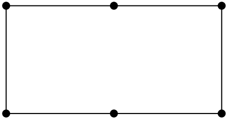
**Poll:** How many edges does $K_{10}$ have? (*Hint:* What is the degree of each vertex in $K_{10}$?)

(A) 20 (B) 45
(C) 50 (D) 90 Answer: (B).

(Each vertex has degree 9, so $2|E| = 10 \times 9$ by Theorem 1.)

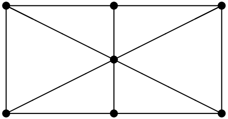
For $n \geq 3$, the $n$-cycle, $C_n$:

$C_6$



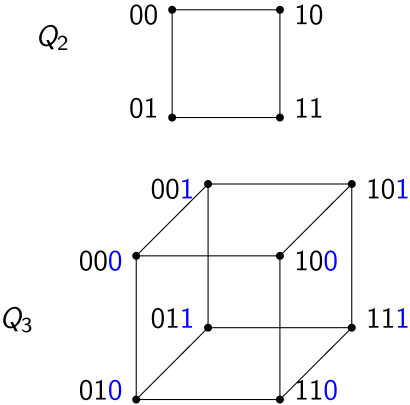
Adding a "central vertex" and $n$ "spokes" to the $n$-cycle gives the $n$-wheel $W_n$:

$W_6$



**Poll:** Which of the following is False?
(A) $C_{20}$ has 20 edges
(B) $W_{20}$ has 40 edges
(C) $W_{20}$ has 20 vertices
(D) $W_{20}$ has a vertex of degree 20
Answer: (C). $W_{20}$ has 21 vertices.

For $n \in \mathbb{Z}^+$, the $n$-cube $Q_n$ is a graph with $2^n$ vertices, each one corresponding to a bit string of length $n$, and edge joining two vertices (strings) if they differ in exactly one bit position:

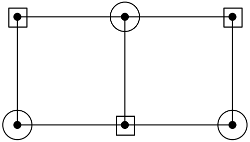



See text for more discussion.

### 10.2.4 Bipartite graphs

**Definition:** Let $G = (V, E)$ be a simple (undirected) graph.
$(i)$ A bipartition is a partition of $V$ into two (disjoint nonempty) sets $V_1$ and $V_2$ such that every edge of $G$ has one endpoint in $V_1$ and the other endpoint in $V_2$.
$(ii)$ If $G$ has a bipartition, then we say that $G$ is bipartite.

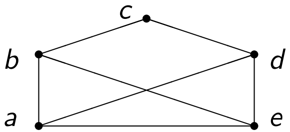
Example: Is this graph bipartite?



YES

$V_1$ : □ $V_2$: ○

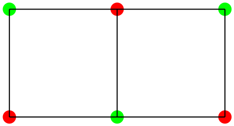
Example: Is this graph bipartite?




NO

**Proof:** Suppose $V_1$ and $V_2$ is a bipartition. Without loss of generality, assume $a \in V_1$. Since $b$ and $e$ are neighbours of $a$, they are both in $V_2$. But an edge connects $b$ and $e$. Contradiction.

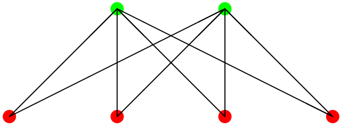
Another way to ask whether a graph is bipartite:
Is it possible to colour each vertex in one of two colours (say, red and green) so that every edge joins a red vertex to a green vertex?



This is phrased formally in Theorem 4 (p. 690).

One more family of graphs:
For $m, n \in \mathbb{Z}^+$, the complete bipartite graph $K_{m,n}$ is a bipartite graph with a bipartition $\{V_1, V_2\}$ such that $|V_1| = m$, $|V_2| = n$, and every vertex of $V_1$ adjacent to every vertex of $V_2$.

$K_{2,4}$



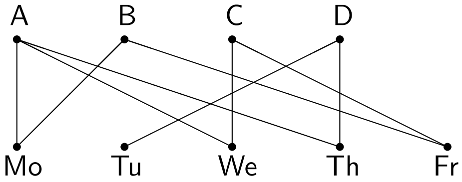
## Bipartite matchings

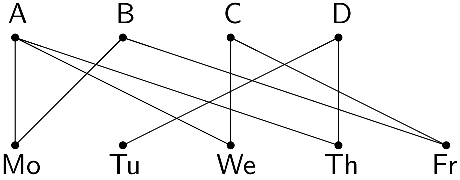
Example: You want to take four one-day courses: Algorithms, Blockchain, Cryptography, and Data Analytics. For each course, you have the choice of at least two days when it is offered (you only need one day for each course):

| | Mon | Tue | Wed | Thu | Fri |
|---|---|---|---|---|---|
| Alg | x | | x | x | |
| Block | x | | | | x |
| Crypto | | | x | | x |
| Data An | | x | | x | |

(x means that you could take the course on that day)

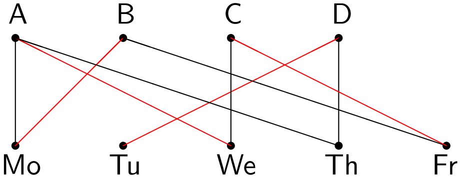
Is it possible for you to take each course? We can set this up as a problem on a bipartite graph:





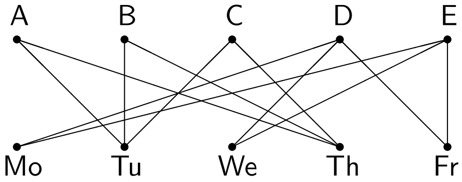

Each edge corresponds to one course held on one day.

Our question is: Can you match the four courses to four different days? (E.g., We can't take both A and B on Monday.) Equivalently, can you choose four edges that do not share any endpoints? In this case, yes: for example,




**Definition:** A matching is a set of edges that do not share any endpoints. The above set of red edges is an example of a matching.

Here is a different schedule, for five courses. Does it have a matching of five edges?




No. Is there an easy way to prove it? Yes: Observe that A, B, and C are only connected to Tu and Th. And you can't take three courses in two days!

Using the graph terminology of the neighbourhood of a set, $N(\{\text{A,B,C}\}) = \{\text{Tu,Th}\}$. The problem, then, is that we have a set whose neighbourhood has fewer vertices than the set itself: $|N(S)| < |S|$ for $S = \{\text{A,B,C}\}$.
Hall's Theorem says that something like this must happen if there is no matching from $V_1$ to $V_2$ that includes every vertex of $V_1$ (called a complete matching from $V_1$ to $V_2$).

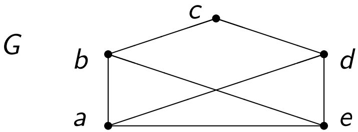
**Hall's "Marriage Theorem":** Let $G = (V, E)$ be a bipartite graph with bipartition $(V_1, V_2)$. There is a complete matching from $V_1$ to $V_2$ *if and only if* $|N(S)| \geq |S|$ for every subset $S$ of $V_1$.

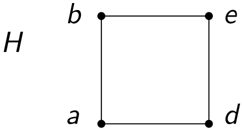
(That is: it is possible to take all the courses *if and only if* no subset of courses is offered on fewer days than there are courses in the subset.)

For large bipartite graphs, there are efficient algorithms that will find a complete matching if one exists exists or else find a set $S$ such that $|N(S)| < |S|$. But this is a story for another course!

## 10.2.7: Subgraphs and other definitions

**Definition:** We say that the graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. And a subgraph $H$ of $G$ is a proper subgraph if $H \neq G$.

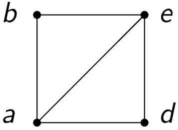



Here, $H$ is a proper subgraph of $G$.

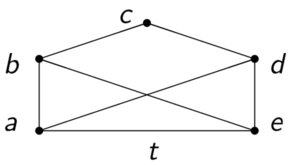
**Definition:** Let $G = (V, E)$ be a simple graph, and let $W \subseteq V$. The subgraph induced by $W$ is the graph $(W, F)$ where $F$ is the set of all edges of $G$ that have both endpoints in $W$.

In the above example, is $H$ the subgraph of $G$ induced by $\{a, b, d, e\}$? No. It also needs the edge with endpoints $a$ and $e$:



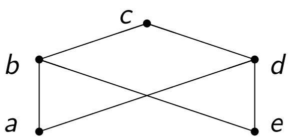
Let $t$ be the edge with endpoints $a$ and $e$ in the graph below.

$G$



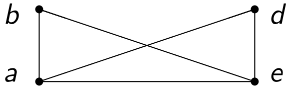

Then $G - t$ is the graph with the edge $t$ removed:

$G - t$

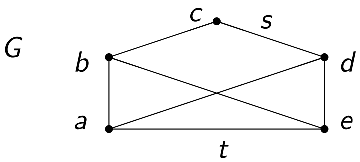



Removing the vertex $c$ from the original graph $G$ also removes all edges incident to $c$. This produces

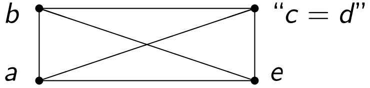
$G - \{c\}$



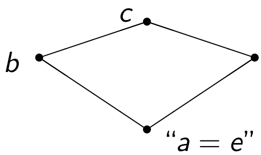
Consider the following graph, with two edges labelled $s$ and $t$.




Then the graph obtained from $G$ by contracting the edge $s$ is

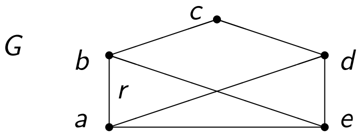



And the graph obtained from $G$ by contracting the edge $t$ is

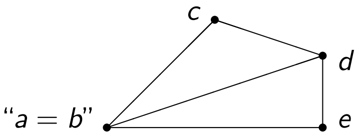



(Notice that double edges are not created; e.g., the edge in $G$ joining $a$ to $b$ gets merged with the edge from $e$ to $b$ into a single edge from the "new" vertex to $b$.)

Now consider the edge $r$ with endpoints $a$ and $b$:




Then the graph obtained from $G$ by contracting the edge $r$ is




See text for more about contraction, as well as unions of graphs and adding edges to graphs.

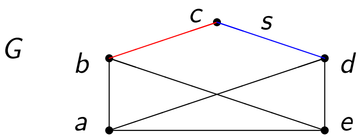
For next Tuesday's class: Read Section 10.3.

The current Connect assignment is due tonight.

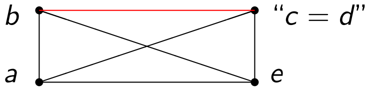
Additional student office hours before Thursday's test:
Wednesday 10:00–11:00 and 3:00–4:00
in my office (S616 Ross) and on Zoom.

SLIDES ADDED, MARCH 26:
These slides may help to clarify the examples of contraction given above by colouring the edges involved.

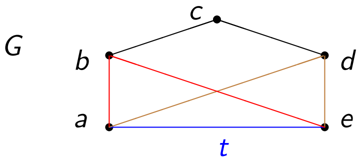
When we contract the edge $s$ in the following graph,



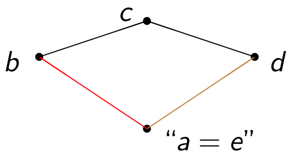

we obtain the graph



When we contract the edge $t$ in the following graph,

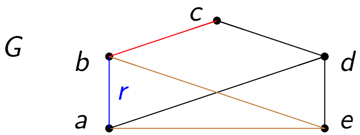



we obtain the graph

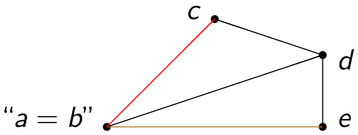



We can think of the contracted edge as shrinking to a point, causing its two endpoints (here $a$ and $e$) to merge into a new single vertex (labelled "$a = e$" in the bottom graph).
The merging of $a$ and $e$ causes the two red edges in $G$ (here, $\{b, a\}$ and $\{b, e\}$) to merge into a single (red) edge whose endpoints are $b$ and the newly created vertex.
Similarly, the two brown edges in $G$ (here, $\{d, a\}$ and $\{d, e\}$) merge into a single (brown) edge whose endpoints are $d$ and the newly created vertex.

When we contract the edge edge $r$ with endpoints $a$ and $b$ in the following graph,




we obtain the graph




Contracting the edge $\{a, b\}$ causing its two endpoints ($a$ and $b$) to merge into a new single vertex (labelled "$a = b$" in the bottom graph).
The merging of $a$ and $b$ causes the red edge $\{b, c\}$ in $G$ to become a (red) edge whose endpoints are the newly created vertex and $c$.
The two brown edges in $G$ ($\{a, e\}$ and $\{b, e\}$) merge into a single (brown) edge whose endpoints are the newly created vertex and $e$.