EECS/MATH 1019
Section 9.5: Equivalence Relations — One more example
Section 10.1: Introduction to Graphs

March 16, 2023

An example for Theorem 2(b) in Section 9.5:

Theorem 2 (p. 645): *(b) Let $\{A_i \mid i \in I\}$ be a partition of A. Define a relation R on A by saying that xRy if and only if x and y are in the same set A_i. Then R is an equivalence relation, and its equivalence classes are the sets A_i.*

Let A be the set of all bit strings of length 8.

Here is a partition on A:
A_1 is the set of all strings in A that end with 0,
A_2 is the set of all strings in A that end with 01, and
A_3 is the set of all strings in A that end with 11.
Then $\{A_1, A_2, A_3\}$ is a partition because each string is in exactly one of these three sets.
Let R be the relation on A defined by saying that string x is related to string y if x and y are both in A_1, or both in A_2, or both in A_3.
E.g. $(11110100)R(01101110)$ (both in A_1)
and $(11110011)R(01110111)$ (both in A_3)
but $\neg(11110101)R(01110111)$ (one in A_2, one in A_3)

Theorem 2(b) tells us that R is an equivalence relation.

Partition of A, the set of bit strings of length 8:

$A_1: \quad \_\_\_\_\_\_\_\,0 \qquad A_2: \quad \_\_\_\_\_\_\,1\,1 \qquad A_3: \quad \_\_\_\_\_\_\,0\,1$

String x is related by R to string y if x and y are both in the same A_i.

I.e., xRy if x and y both have 0 for their last bit, or if x and y agree in the last two bits.

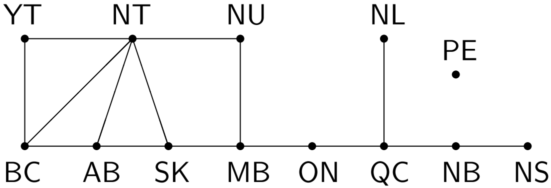
We can write $R = R_1 \cup R_2$, where R_1 and R_2 are the following relations on A:
xR_1y if x and y both end with 0, and
xR_2y if x and y have the same last two bits.

Then R_2 is an equivalence relation. What about R_1?
R_1 is symmetric and transitive but not reflexive, because any string ending with 1 is not related to itself.
In terms of ordered pairs, R_1 is the set of all $(x, y) \in A \times A$ such that x and y both end with 0.

Chapter 10: Graphs

Suppose we are working on something involving the provinces and territories of Canada, but the only information we need to know about them is which ones share a land border with which others. (<u>Remark</u>: This is a symmetric relation on the set of provinces and territories.)

This information can be summarized in a **graph**.


YT
NT
NU
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NB
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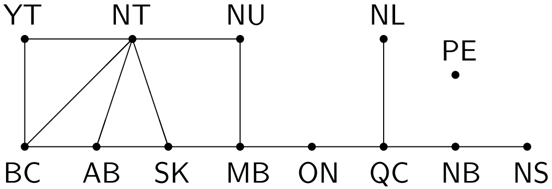


This is an example of a graph with 13 vertices (also called nodes) and 15 edges.
("Vertices" is the plural of "vertex.")

We often write V to denote the set of vertices. In this example, $V = \{$YT, NT, NU, NL, BC,$\ldots$, PE, NB, NS$\}$, corresponding to the set of provinces and territories.
We often write E for the set of edges. Each edge connects two vertices, which we call the endpoints of the edge. We have one edge connecting YT and BC; another connecting BC and AB; and so on. Here, each edge corresponds to a pair of provinces/territories that share a land border.

Some other examples of graphs:

Example 1: Social networks: Here, each vertex is a person, and each edge represents some kind of association between a pair of people, such as friendship, or kinship,
or going to some of the same parties,
or regularly exchanging email at work,
or having children at the same school,. . .

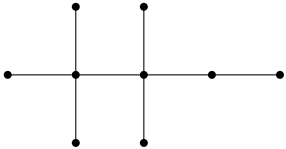
Some researchers investigate the structure of social networks:

Do they break into a few large groups, with not many edges between different groups?

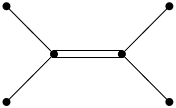
Are there a few vertices that have many more edges than others?

How might information (or a transmissible disease) propagate through such a network?

Example 2: Molecules:



ethanol C_2H_5OH

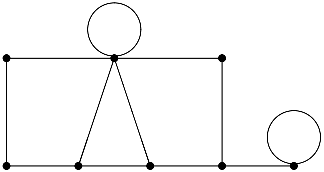


ethane C_2H_4

Each vertex is an atom (carbon, oxygen, or hydrogen in these examples), and each edge represents a chemical bond between two atoms. The ethane molecule has a double bond between the two carbon atoms, which we can represent by two edges connecting the same pair of vertices.

In general, a graph can have multiple edges between a pair of vertices. We often use the word multigraph to describe such a graph.

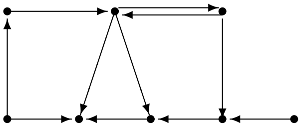
A graph can also have “loops.” A loop is an edge whose two endpoints are the same vertex. For example, this graph has two loops:



A simple graph is a graph with no loops and no multiple edges. Most of the graphs that we shall consider will be simple graphs.

Example 3: The Internet   The vertices are web pages.

We represent a hyperlink from one page that points to another page by an edge. But the direction is important. We need directed edges (or arcs) to represent hyperlinks.



A directed graph or digraph (V, E) consists of a set of vertices V and a set of directed edges E.
Each directed edge is associated with an ordered pair of vertices (u, v), and we say that this edge starts at u and ends at v.

We can have an edge from u to v and another edge from v to u (as in above diagram).
We can also have multiple directed edges (directed multigraph), and some edges directed and others undirected (mixed graph), but these kinds of graph are used less often.

Example 4: International trade models in economics:

The vertices are countries, and an edge from country A to country B represents trade flow from A to B. It is natural here to attach an amount to each edge (say in millions of dollars per year).

For example, see L. Kemper and T. Plümper, Journal of Social Structure, vol 4, No. 1 (2003).
www.cmu.edu/joss/content/articles/volume4/KrempelPlumper.html

Example 5: Contact tracing for disease spread

The vertices are people, and there is a directed edge from person u to person v if the public health investigators believe that u transmitted the disease to v.

Here is an article that discusses what properties contact tracing graphs typically have:
K. Nagarajan et al., "Social Network analysis method for exploring SARS-CoV-w contact tracing data." *BMC Medical Research Methodology* vol. 20, article number 233 (2020).
doi.org/10.1186/s12874-020-01119-3

Example 6: Precedence graphs: Used in concurrent processing (see p. 679), and also in project scheduling.

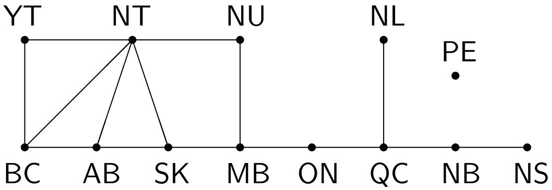
A company is ready to make a new product, which is made of two parts. Several tasks must be done, and some must precede others:

Activity	Predecessors	Duration (days)
A = Train workers	—	5–7
B = Buy raw materials	—	8–10
C = Make Part 1	A, B	8–9
D = Make Part 2	A, B	5–7
E = Test Part 2	D	9–11
F = Assemble Parts 1 and 2	C, E	10–12

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & C \arrow[r] & F \\
B \arrow[r] \arrow[ur] & D \arrow[r] & E \arrow[u]
\end{tikzcd}$$

(How long should the whole process take?)

More terminology for graphs (subsection 10.2.2)



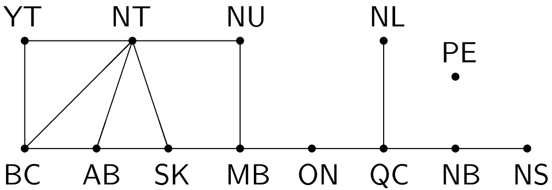
For <u>undirected</u> graphs:

Suppose an edge e has endpoints u and v. Then we say that e is incident with u and with v, that u and v are adjacent, and that u and v are neighbours.

The set of all neighbours of a vertex v is called the neighbourhood of v, and is denoted $N(v)$.

E.g. The neighbourhood of QC is { ON, NL, NB }.

The neighbourhood of PE is the empty set.



If $A \subseteq V$, then $N(A)$ is the set of all the neighbours of the vertices of A. That is, $N(A) = \bigcup_{v \in A} N(v)$.

E.g. Let W be the set of the Western provinces: $A = \{$ BC, AB, SK, MB $\}$. Then $N(W)$ is $\{$ YT, NT, NU, ON, BC, AB, SK, MB $\}$.

In an undirected graph with no loops, the degree of a vertex is the number of edges incident with v. We write it $\deg(v)$.

Poll: Which is FALSE for the above graph?

(A) It has two vertices of degree 1.

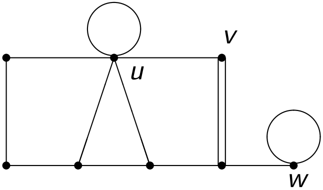
(B) It has four vertices of degree 2.

(C) It has four vertices of degree 3.

(D) It has one vertex of degree 5.

<u>Answer</u>: (C). There are five vertices of degree 3.

In an undirected graph with loops, the degree of a vertex is the number of edges incident with v, but we count each loop twice.



In this graph, $\deg(w) = 3$, $\deg(u) = 6$, and $\deg(v) = 3$.

One reason that we treat loops like this is because it allows us to obtain the following neat fact:

Theorem 1: Let G be an undirected graph with m edges. Then the sum of the degrees of all of the vertices is equal to $2m$.

E.g. in the above graph, $m = |E| = 13$, and the sum of the degrees is $2 + 6 + 3 + 2 + 3 + 3 + 4 + 3 = 26$.

Idea of proof: count the number of  . ("half-edges")

Next class: Read Section 10.2.

The second midterm test will be on Thursday March 23, in this room. The test will focus on Sections 3.2, 5.1, 5.2, 5.3, 8.1, 8.2, 9.1, and 9.5. Some subsections are omitted (see eClass page for details).

Homework updates:
- Problem Set C is due tonight (in Crowdmark).
- Homework assignment 7 (in Connect) is due Tuesday March 21.