EECS/MATH 1019
Section 9.1 and 9.5: Relations

March 14, 2023

Recap:

Definition: (Section 9.1.1) *Let A and B be sets. A relation R from A to B is a subset of $A \times B$.*

Example (1): "speaks" is a relation from the set of people to the set of languages. This subset of $\{$people$\} \times \{$languages$\}$ includes the ordered pair E.g., (J Trudeau, French).

We write xRy to mean that x is related by R to y, i.e. that $(x, y) \in R$.

When A and B are the same set, we often say that "R is a relation on A" rather than "R is a relation from A to A." In this case, R is a subset of $A \times A$.

Example (2) "is greater than" is a relation on $\mathbb{R}$.

For the rest of today, we'll just consider relations on a set (i.e., from a set to itself.)

Examples: (3) "divides" on $\mathbb{Z}$: m divides n if $m \neq 0$ and $\frac{n}{m} \in \mathbb{Z}$.
We also write this as $m \mid n$.

(4) "speaks a language in common with" on the set of all people

(5) $2x^3 + y^2 = 3xy$. This equation defines a relation E on $\mathbb{R}$ by saying that xEy if and only if the values for x and y satisfy this equation.

Here are three properties that a relation on a set could have:

- Reflexive
- Symmetric
- Transitive

Definition: The relation R on A is *symmetric* if

$$\forall x \, \forall y \, (xRy \ \rightarrow \ yRx) \, .$$

Which of the following relations are symmetric?
"Less than": $<$ on $\mathbb{R}$: No
"Equals": $=$ (on any set): Yes

(A) "Not equal": $\neq$ (on any set): Yes
(B) "Speaks a common language with": Yes
(C) "Has the same length" on a set of strings Σ^* Yes
(D) "divides": $x|y$ on $\mathbb{Z}^+$ No
(E) "Born in same country as" on the set of people: Yes

Remark: If R is symmetric, then $\forall x \, \forall y \, (xRy \leftrightarrow yRx)$.

Remark: In fact, the relation "$\leq$" on $\mathbb{R}$ is *antisymmetric*.

Definition: The relation R is antisymmetric if

$$\forall a \forall b\, ([aRb \text{ and } bRa] \ \rightarrow\ a = b)$$

Equivalently, R is antisymmetric if

$$\forall a \forall b\, ([(a, b) \in R \ \text{ and } \ (b, a) \in R] \ \rightarrow\ a = b)$$

For "$\leq$", this says: *If $a \leq b$ and $b \leq a$, then $a = b$.* (Always true.)

Is the relation "$<$" on $\mathbb{R}$ also antisymmetric?
In this case, $(a < b) \wedge (b < a)$ is always False.
So $[(a < b) \wedge (b < a)] \ \rightarrow\ a = b$ is always True.
Therefore "$<$" is antisymmetric.

Exercise: Show that the relation "divides" on $\mathbb{Z}^+$ is also antisymmetric.

Definition: The relation R on A is *reflexive* if

$$\forall x \in A \, (xRx) \, .$$

Which of the following relations are reflexive?
"Less than": $<$ on $\mathbb{R}$: No
"Less than or equal to": $\leq$ on $\mathbb{R}$: Yes

Poll: Which if these is (are) NOT reflexive?
(A) "Equals": $=$ (on any set)
(B) "Not equal": $\neq$ (on any set)
(C) "Speaks a common language with"
(D) "Has the same length" on a set of strings Σ^*
(E) "divides": $x|y$ on $\mathbb{Z}^+$
(F) "Born in same country as" on the set of people

Answer: (A), (C), (D), (E), (F) are reflexive, but (B) is not.

More details about the reflexive property:

- "Less than": $<$ on $\mathbb{R}$: This is not reflexive: in fact, $x < x$ is false for every real x
- "Less than or equal to": $\leq$ on $\mathbb{R}$: This is reflexive because $x \leq x$ for every real x

(A) "Equals": $=$ (on any set): This is reflexive, since $x = x$ for every x

(B) "Not equal": $\neq$ (on any set): This is not reflexive, since it is always false that $x \neq x$

(C) "Speaks a common language with": This is reflexive, since each person speaks a common language with themself

(F) "Born in same country as": This is reflexive, since each person was born in the same country as themself

Definition: The relation R on A is *transitive* if

$$\forall x\, \forall y\, \forall z\, ([xRy \text{ and } yRz] \rightarrow xRz)\,.$$

Which of the following relations are transitive?
"Less than": $<$ on $\mathbb{R}$: Yes (also $\leq$)
"Equals": $=$ (on any set): Yes

Poll: Which if these is (are) NOT transitive?
(A) "Not equal": $\neq$ (on any set)
(B) "Speaks a common language with"
(C) "Has the same length" on a set of strings Σ^*
(D) "divides": $x|y$ on $\mathbb{Z}^+$
(E) "Born in same country as"

Answer: (C), (D), (E) are transitive, but (A), (B) are not.

Definition: The relation R on A is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Are the following equivalence relations?
"Less than or equal to": $\leq$ on $\mathbb{R}$: No (not symmetric)
"Equals": $=$ (on any set): Yes
"Not equal": $\neq$ (on any set): No (neither reflexive nor transitive)
"Speaks a common language with": No (not transitive)
"Has the same length" on a set of strings Σ^*: Yes
"divides": $x|y$ on $\mathbb{Z}^+$: No (not symmetric)
"Born in same country as": Yes

Definition: The relation R on A is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example (6): Define the relation C_2 on $\mathbb{Z}$ by

$$x \, C_2 \, y \quad \text{if and only if} \quad y - x = 2k \quad \text{for some integer } k$$

$$\text{i.e.,} \quad \text{if and only if} \quad \frac{y-x}{2} \text{ is an integer.}$$

This is an equivalence relation (exercise), which is also called "x is congruent to y modulo 2."
Observe that $x \, C_2 \, y$ if and only if x and y are both even or both odd. That is, if and only of x and y have the "same parity."

(7) Let A be the set of all people. Define the relation K on A as follows: For two people u and v, we say $u\,K\,v$ if u and v were born within 1 km of each other.

Poll: Is K an equivalence relation?

(A) Yes
(B) No — it is not reflexive
(C) No — it is not symmetric
(D) No — it is not transitive.

<u>Answer</u>: (D)

Equivalence classes

Definition: Let $\sim$ be an equivalence relation on a (nonempty) set A. Let $x \in A$. The *equivalence class of* x (determined by the relation $\sim$) is the set of elements of A that are related to x:

$$[x] = \{ y \in A \mid y \sim x \}.$$

Examples

(a) For the relation "Same length as" on the binary strings $\{0,1\}^*$, the equivalence class [01] is

$$[01] = \{01, 10, 11, 00\}.$$

(b) For the relation "Born in the same country as" on the set of people, the equivalence class [Justin Trudeau] is the set of all people born in Canada.

(c) For "Same parity" on $\mathbb{Z}$ $\left(x\, C_2\, y \leftrightarrow \frac{y-x}{2} \in \mathbb{Z}\right)$, the equivalence class [7] is the set of all odd integers.

<u>Some observations</u>:

(a) For the relation "Same parity",

$$\begin{aligned} [7] &= \{y \in \mathbb{Z} \mid y - 7 \text{ is even}\} \\ &= [1] = [3] = [39] \qquad \ldots \text{and so on.} \end{aligned}$$

(b) For the relation "Born in the same country as" on the set of people,

$$[\text{Justin Trudeau}] = [\text{Justin Bieber}] = [\text{Drake}] \qquad \ldots \text{and so on.}$$

This suggests that the following is true:

Theorem 1 (page 643) (slightly rephrased):
Let $\sim$ be an equivalence relation on the nonempty set A. For all $x, y \in A$:
(a) if $x \sim y$ if and only if $[x] = [y]$;
(b) if $x \not\sim y$, then $[x]$ and $[y]$ are disjoint.

(See text for proof.)

Back to our examples:
(a) For the relation "Same parity" on $\mathbb{Z}$:

$$\begin{aligned}
[1] &= \{\dots, -3, -1, 1, , 3, 5, 7, \dots\} \\
[1] &= [3] = [5] = [7] = [-1] = [-3] = \dots \\
[2] &= \{\dots, -4, -2, 0, 2, 4, 6, \dots\} \\
[2] &= [4] = [6] = [0] = [-2] = \dots
\end{aligned}$$

(b) For the relation "Born in the same country as" on the set of people,

[Justin Trudeau] = [Justin Bieber] = [Drake] = ...

Also, [B Obama] = [J Biden] = [Serena Williams] = ...

For the relation (b) "Born in the same country as":
All the people who were born in Canada are all related to each other, and are all in one equivalence class.
Similarly, all the people born in the USA are related to each other, and are in another equivalence class.
This relation breaks the set of all people into several equivalence classes, one for each country, consisting of all the people who were born in that country. In fact, these sets form a *partition* of the set of all people.

Definition: *Let A be a set. A partition of A is a collection of sets $A_1, A_2, \ldots$ such that each A_i is a nonempty subset of A, and each element of A is contained in exactly one of the A_i's.*

Equivalently:
Definition: *Let A be a set. A partition of A is a collection of sets $A_1, A_2, \ldots$ of nonempty disjoint subsets of A, and whose union is A.*

For the relation "Same parity" on $\mathbb{Z}$, there are two equivalence classes:
A_1 = the set of odd integers ($= [1]$), and
A_2 = the set of even integers ($= [2]$).

These two equivalence classes forms a partition of the set of integers.

For the equivalence relation "Same length" on the set of strings Σ^*, the equivalence classes can be written as A_i is the set of all strings of length i ($i = 0, 1, 2, \ldots$)

Remark: If A is an uncountable set, then there could be uncountably many equivalence classes. Then we could not list them as $A_1, A_2, \ldots$ That is why the text writes the collection of equivalence classes as A_i for i taking values in some index set I (which may be uncountable).

Theorem 2 (p. 645): *(a) Let $\sim$ be an equivalence relation on a nonempty set A. Then the collection of all equivalence classes determined by $\sim$ is a partition of the set A.*
(b) Let $\{A_i \mid i \in I\}$ be a partition of A. Define a relation R on A by saying that xRy if and only if x and y are in the same set A_i. Then R is an equivalence relation, and its equivalence classes are the sets A_i.

E.g. Consider the partition $A_0, A_1, A_2, \ldots, A_9$ on $\mathbb{Z}^+$, defined so that A_i is the set all numbers whose rightmost digit is i.
This partition defines a relation: x is related to y if they have the same rightmost digit. This is an equivalence relation.

E.g. The set of all currently active professional hockey players is partitioned according to which team they are playing for. The corresponding equivalence relation has two players related if they play on the same team.

Section 9.1.5: Combining Relations

Definition: *Let A and B be sets. A relation from A to B is a subset of $A \times B$.*

Example: "speaks" is a relation from the set of people to the set of languages. Let's call this relation "S." So

$$S = \{(p, \ell) \mid p \text{ is a person who speaks language } \ell \}.$$

Also, let R be the relation "reads" from the set of people to the set of languages:

$$R = \{(p, \ell) \mid p \text{ is a person who reads language } \ell \}.$$

Then $S \cap R$ is the set of (p, ℓ) such that p speaks and reads ℓ.

And $S - R$ is the set of (p, ℓ) such that p speaks ℓ but does not read it.

And the complement $\overline{S}$ is the set of (p, ℓ) such that p does not speak language ℓ.

Similarly for $S \cup R$ and $R - S$ and $R \oplus S$.
(We won't discuss the "composite" of S and R.)

Next class: Read Section 10.1.

The second midterm test will on Thursday March 23. It will focus on Sections 3.2, 5.1, 5.2, 5.3, 8.1, 8.2, 9.1, and 9.5. Some subsections are omitted (see eClass page for details).

Homework updates:

- Problem Set C is due Thursday March 16.
- Homework assignment 7 (in Connect) is due <u>Tuesday</u> March 21.