EECS/MATH 1019
Section 5.1: Mathematical Induction – Part I

February 9, 2023

Warm-up Polls: Suppose p, q, r, s, and t are propositions. And suppose we know that all of the following are true:

$p \to q$
$q \to r$
$r \to s$
$s \to t$

And finallly, suppose we also know that p is true. Then what can we conclude about the truth of q, r, s, and t?

(A) q is true, but we don't know about the others
(B) t is true, but we don't know about the others
(C) q and t are true, but we don't know about the others.
(D) q and r are true, but we don't know about the others
(E) q, r, s, and t are all true.

Answer: (E). (Use Modus ponens 4 times)

Second question: Suppose p, q, r, s, and t are propositions. And suppose we know that all of the following are true:

$p \rightarrow q$

$q \rightarrow r$

$r \rightarrow s$

$s \rightarrow t$

Poll: With only the above information, what can we conclude about the truth of q, r, s, and t?

(A) t is true, but we don't know about the others
(B) q, r, s, and t are all true.
(C) q, r, s, and t are all false.
(D) We cannot conclude whether any of them are true or false.

Answer: (D). (It is possible that they are all true, or all false, or maybe some other combination)

Third question: Suppose p, q, r, s, and t are propositions. And suppose we know that all of the following are true:

$p \to q$

$q \to r$

$r \to s$

$s \to t$

Poll: Suppose that we also know that r is true. The what can we conclude about the truth of p, q, s, and t?

(A) s is true, but we don't know about the others
(B) t is true, but we don't know about the others
(C) s and t are true, but we don't know about the others.
(D) s and t are true, and p and q are false
(E) p, q, s, and t are all true.
(F) We cannot conclude whether any of them are true or false.

Answer: (C). (Use Modus ponens twice. But that's all we can do.)

We can generalize to the following situation.
Let $P(1), P(2), P(3), P(4) \ldots$ be statements.
Suppose we know that all the following are true:

$$P(1) \to P(2)$$
$$P(2) \to P(3)$$
$$P(3) \to P(4)$$
$$\vdots$$

At this point, we don't know much about whether each $P(k)$ is true or false.

However, if we also know that $P(1)$ is true, then we conclude that $P(2)$ is true and $P(3)$ is true and $P(4)$ is true...

This applies even if there are infinitely many statements, i.e. a statement $P(k)$ for every positive integer k, and we know $P(k) \to P(k+1)$ for every positive integer k.
If we also know that $P(1)$ is true, then we can conclude that $P(k)$ is true for every positive integer k.

This is called the Principle of Mathematical Induction.

Principle of Mathematical Induction

Let $P(1), P(2), P(3), \ldots$ *be statements. Assume*

(a) $P(1)$ *is true, and*
(b) $\forall k \in \mathbb{Z}^+ \,(P(k) \to P(k+1))$.

Then $P(n)$ *is true for every* $n \in \mathbb{Z}^+$.

Example 1: Prove $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$.

(E.g. for $n = 6$, we have $1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \cdot 7}{2}$.)

For each $n \in \mathbb{Z}^+$, let $P(n)$ be the statement
"$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$."
We want to prove $P(n)$ for every $n \in \mathbb{Z}^+$.

(a) $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$, which is **true**.

Principle of Mathematical Induction

Let $P(1), P(2), P(3), \ldots$ be statements. Assume
(a) $P(1)$ is true, and (b) $\forall k \in \mathbb{Z}^+ \, (P(k) \to P(k+1))$.
Then $P(n)$ is true for every $n \in \mathbb{Z}^+$.

In Example 1: For each $n \in \mathbb{Z}^+$, let $P(n)$ be the statement "$1+2+3+\ldots+n = \frac{n(n+1)}{2}$". We want to prove $\forall n \in \mathbb{Z}^+ \, (P(n))$.

(a) $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$, which is **true**.

(b) $P(k+1)$ says $1+2+3+\ldots+k+(k+1) = \frac{(k+1)(k+2)}{2}$.
We need to prove $\forall k \in \mathbb{Z}^+ \, (P(k) \to P(k+1))$. Here's how.
Let $k \in \mathbb{Z}^+$ and assume that $P(k)$ is true. That is, assume

$$1+2+3+\ldots+k = \frac{k(k+1)}{2}. \quad \text{Then}$$

$$\begin{aligned} 1+2+3+\ldots+k+(k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \left(\frac{k}{2}+1\right)(k+1) = \frac{(k+2)(k+1)}{2}. \end{aligned}$$

This says that $P(k+1)$ is true! So we have proved that (b) is true.
By mathematical induction, we conclude that $P(n)$ is true for every $n \in \mathbb{Z}^+$. This completes Example 1.

Example 2: Use mathematical induction to prove that $7^n - 4^n$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

Before we proceed, let's review the definition of "divisible."

Let b and d be integers, with $d \neq 0$.

We say that b is divisible by d if (and only if) b/d is an integer.

Equivalently, b is divisible by d if and only if there exists an integer w such that $b = dw$.

E.g. 65 is divisible by 5 because $65/5 = 13$, which is an integer. Alternatively, 65 is divisible by 5 because $65 = 5 \times 13$ (and 13 is an integer).

Other terminology:
5 is a divisor of 65
5 is a factor of 65
65 is a multiple of 5

Principle of Mathematical Induction

Let $P(1), P(2), P(3), \ldots$ be statements. Assume
(a) $P(1)$ is true, and
(b) $\forall k \in \mathbb{Z}^+ (P(k) \to P(k+1))$.
Then $P(n)$ is true for every $n \in \mathbb{Z}^+$.

Example 2: Use mathematical induction to prove that $7^n - 4^n$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

For each $n \in \mathbb{Z}^+$, let $P(n)$ be the statement $7^n - 4^n$ is divisible by 3.

(Quick check:

$n = 1$: $7^1 - 4^1 = 3$ $\quad\therefore P(1)$ is true.

$n = 2$: $7^2 - 4^2 = 49 - 16 = 33$ $\quad\therefore P(2)$ is true.

$n = 3$: $7^3 - 4^3 = 343 - 64 = 279 = 93 \times 3$ $\quad\therefore P(3)$ is true.

Looks okay so far!)

So far, we have verified (a) (often called the *base case*).
Now we need to prove (b) (often called the *inductive step*).

Principle of Mathematical Induction *Assume*
(a) $P(1)$ is true, and (b) $\forall k \in \mathbb{Z}^+ (P(k) \to P(k+1))$.
Then $P(n)$ is true for every $n \in \mathbb{Z}^+$.

Example 2 (continued): For each $n \in \mathbb{Z}^+$, let $P(n)$ be the statement $7^n - 4^n$ is divisible by 3.
We have shown (a). Now we need to prove (b).
Let $k \in \mathbb{N}$. We have

$P(k)$: $7^k - 4^k$ is divisible by 3

$P(k+1)$: $7^{k+1} - 4^{k+1}$ is divisible by 3

Assume that $P(k)$ is true. Then $7^k - 4^k = 3t$ for some $t \in \mathbb{Z}$.
(Note that t depends on k. That's okay.) Then

$$\begin{aligned} 7^{k+1} - 4^{k+1} &= 7(7^k) - 4^{k+1} = 7(4^k + 3t) - 4^{k+1} \\ &= 7(4^k) + 7(3t) - 4(4^k) \\ &= (7-4)(4^k) + 7(3t) = 3(4^k + 7t). \end{aligned}$$

Hence $7^{k+1} - 4^{k+1}$ is divisible by 3. We conclude that (b) is true.
Therefore, by induction, $\boxed{P(n) \text{ is true for every } n \in \mathbb{Z}^+.}$ Q.E.D.

We have proved that $7^n - 4^n$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

Example 3 (Geometric sums): Fix a real number r, with $r \neq 1$. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the statement

$$1 + r + r^2 + \cdots + r^n \;=\; \frac{r^{n+1} - 1}{r - 1}\,.$$

We can use mathematical induction to prove that $P(n)$ is true for every $n \in \mathbb{Z}^+$.
See Examples 3 and 4 in the text (pages 339–340) for details.

We can also use induction to prove inequalities.

Example 4 (Bernoulli's Inequality):
Let v be a real number such that $v > -1$. Then

$$(1+v)^n \;\geq\; 1+nv \qquad \text{for all } n \in \mathbb{Z}^+.$$

Proof: Fix $v \in (-1, \infty)$.
For $n \in \mathbb{Z}^+$, let $P(n)$ be the statement $(1+v)^n \;\geq\; 1+nv$.
(a) $n=1$: $P(1)$ says $(1+v) \;\geq\; 1+v$. This is obviously true.
(b) Let $k \in \mathbb{Z}^+$ and assume $P(k)$ is true, i.e. $(1+v)^k \;\geq\; 1+kv$.
(We want to deduce $P(k+1)$: $(1+v)^{k+1} \;\geq\; 1+(k+1)v$.)

$$\begin{aligned}
(1+v)^{k+1} &= (1+v)^k(1+v) \\
&\geq (1+kv)(1+v) && \text{(by } P(k) \text{ and } v > -1) \\
&= 1+kv+v+kv^2 \\
&= 1+(k+1)v+kv^2 && \text{Now, use fact that } kv^2 \geq 0\text{:} \\
&\geq 1+(k+1)v+0\,. && \text{This shows that } P(k+1) \text{ is true!}
\end{aligned}$$

We have proved $P(1)$ and $\forall k \in \mathbb{Z}^+ \, (P(k) \rightarrow P(k+1))$.
Therefore $P(n)$ holds for every $n \in \mathbb{Z}^+$. Q.E.D.

Example 5: For $n \in \mathbb{Z}^+$, let $P(n)$ be the statement $7n^2 \leq 4^n$.

We shall prove that for every $k \in \mathbb{Z}^+$, $P(k) \to P(k+1)$.

(Note that $P(k+1)$ says $7(k+1)^2 \leq 4^{k+1}$.)

Assume that $P(k)$ is true, i.e. $7k^2 \leq 4^k$. Then

$$\begin{aligned}
7(k+1)^2 &= 7(k^2+2k+1) \\
&\leq 7(k^2+2k^2+k^2) \qquad (\text{since } 1 \leq k, \text{ so } k \leq k^2) \\
&= 4 \times 7 \times k^2 \qquad \text{Now apply } P(k)\text{:} \\
&\leq 4 \times 4^k.
\end{aligned}$$

So we have shown $7(k+1)^2 \leq 4^{k+1}$, which is $P(k+1)$.

We have proved $(\forall k \in \mathbb{Z}^+)\,(P(k) \to P(k+1))$.

Check $P(1)$: $7(1)^2 \leq 4^1$ False!

Check $P(2)$: $7(2)^2 \leq 4^2$ False

Check $P(3)$: $7(3)^2 \leq 4^3$ True!

What can we conclude? Since $P(3)$ and $P(3) \to P(4)$ are both true, $P(4)$ is true. And $P(4) \to P(5)$ is true, so $P(5)$ is true.

Conclusion: $P(n)$ is true for every $n \geq 3$.

Modified Principle of Mathematical Induction

Let $M \in \mathbb{Z}$, and let $P(M), P(M+1), P(M+2), \ldots$ be statements. Assume

(a') $P(M)$ is true, and
(b') $P(k) \to P(k+1)$ for every $k \in \mathbb{Z}$ such that $k \geq M$.

Then $P(n)$ is true for every $n \in \mathbb{Z}$ such that $n \geq M$.

(Note: Assumption (b') says that $P(M) \to P(M+1)$ and $P(M+1) \to P(M+2)$ and $P(M+2) \to P(M+3)$ and ....)

Example 6: Prove the following formula from Section 2.4:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $P(n)$ be the statement $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.
Check the base case $n = 1$:

$$\sum_{i=1}^{1} j^2 = 1^2 = 1, \quad \text{and} \ \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1.$$

Therefore $P(1)$ is true.
We don't need it for induction, but let's check the case $n = 2$ also:

$$\sum_{i=1}^{2} j^2 = 1^2+2^2 = 5, \quad \text{and} \ \frac{2(2+1)(2(2)+1)}{6} = \frac{2(3)(5)}{6} = 5.$$

So $P(2)$ is also true.

Now we'll show $\forall k \in \mathbb{Z}^+ \, (P(k) \to P(k+1))$.

Let k be a positive integer and assume $P(k)$ is true, i.e.

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}.$$

We want to deduce $P(k+1)$, i.e.

$$\sum_{j=1}^{k+1} j^2 \stackrel{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

which is the same as

$$\left(\sum_{j=1}^{k} j^2\right) + (k+1)^2 \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}.$$

By $P(k)$, the above is the same as

$$\left(\frac{k(k+1)(2k+1)}{6}\right) + (k+1)^2 \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}.$$

So we only need to check

$$
\begin{aligned}
\left(\frac{k(k+1)(2k+1)}{6}\right)+(k+1)^2 &\overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}. \\
\leftrightarrow \quad \frac{k(k+1)(2k+1)+6(k+1)^2}{6} &\overset{?}{=} \frac{(k+1)(k+2)(2k+3)}{6}. \\
\leftrightarrow \quad \frac{(k+1)[k(2k+1)+6(k+1)]}{6} &\overset{?}{=} \frac{(k+1)(2k^2+7k+6)}{6} \\
\leftrightarrow \quad \frac{(k+1)[2k^2+k+6k+6]}{6} &\overset{?}{=} \frac{(k+1)(2k^2+7k+6)}{6}
\end{aligned}
$$

which is an equality! This proves that $P(k+1)$ is true.

Thus we proved that $P(k) \to P(k+1)$ for every positive integer k. And since we also showed that $P(1)$ is true, it follows from mathematical induction that $P(n)$ is true for every positive integer n. That is,

$$
\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{for every } n \in \mathbb{Z}^+.
$$

For test next Tuesday:
See description in eClass
Bring your York ID
Don't be late – arrive early if you can

For next Thursday: Finish reading Section 5.1 (omit Example 12), and read Section 5.2 (5.2.4 is optional).

Homework assignment 4 (in Connect) is due Sunday.