EECS/MATH 1019
Section 3.2: Growth of Functions

February 7, 2023

Background: Properties of Absolute Value

The absolute value of a real number x is the nonnegative number $|x|$ defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

E.g. $|-12| = -(-12)$ (since $-12 < 0$)
$= 12 = |12|$.

<u>Properties:</u> For $x, y \in \mathbb{R}$,

(i) $|xy| = |x|\,|y|$.

(ii) $-|x| \leq x \leq |x|$ (e.g. for $x = -12$, this says
$-|-12| \leq -12 \leq |-12|$

(iii) $|x+y| \leq |x| + |y|$ ("Triangle Inequality")
(e.g. $|-12+5| \leq |-12| + |5| = 17$.

(iv) General Triangle Inequality: For real $a_1, a_2, \ldots, a_n$, we have

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Poll: Suppose w is a real number that satisfies $-7 \le w \le 7$. What can we conclude?

(A) $|w| = 7$

(B) $|w| \le 7$

(C) $|w| \ge 7$

(D) We cannot be sure of any of these.

Correct answer: ?????

Growth of Functions

How long does a computer program (or algorithm) take to run when the size of its input goes up by a factor of 10? or 100?

For example, suppose our team has developed a mathematical model to describe the spread of a particular disease in a community (e.g. Covid-19, or Ebola). They write a program that

(a) seeks available data about a community (locations of homes, schools, workplaces; family sizes; travel patterns, etc.) and then

(b) runs a series of simulations to see how the disease might spread after one person is initially infected.

Your goal is to estimate the total number of people who will become infected in one month.

You have tried it out for communities of 50 people, where it takes about 5 minutes to give an estimate of the desired quantity. How long will it take to run on a community of size 500? or 50,000? Or more?

That is, how do you expect your program to "scale up"? We need a bit more information.

In this disease example, suppose that for a community of N people, the program takes $5N$ seconds to perform part (a), and $0.0004N^3$ seconds to do part (b).
Let $f(N)$ be the total running time (in seconds) of the program for a community of size N:

$$f(N) \;=\; 5N \;+\; 0.0004N^3\,.$$

In particular, for $N = 50$, we have $5N = 250$ and $0.004N^3 = 0.0004 \times (50)^3 = 0.0004\times 125{,}000 = 50$, so the number of seconds needed for $N = 50$ is $f(50) = 250 + 50 = 300$, which is 5 minutes.
For $N = 100$:

$$f(100) \;=\; 5 \times 100 \;+\; 0.0004(100)^3 \;=\; 500 \;+\; 400 \;=\; 900,$$

and 900 seconds is 15 minutes.

But as we see on the Excel sheet, as N gets larger, the term $0.0004N^3$ is much bigger than the term $5N$.

Definition: Let f and g be functions with domain $\mathbb{Z}$ or $\mathbb{R}$ (or $\mathbb{Z}^+$ or $\mathbb{R}+$) and co-domain $\mathbb{R}$. We say that $f(x)$ is $O(g(x))$ if there exist constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \text{whenever } x > k\,.$$

Example: We'll use $f(x) = 5x + 0.0004x^3$ again (domain $\mathbb{Z}^+$).
We claim that $f(x)$ is $O(x^3)$. For polynomials like f, it is often convenient to use $k = 1$.
Observe that $x \leq x^3$ whenever $x > 1$ (because when $x > 1$, we have $1 < x^2$ and $x > 0$, so $1 \cdot x < x^2 \cdot x = x^3$).
Therefore, whenever $x > 1$,

$$|f(x)| = 5x + 0.0004x^3 \leq 5x^3 + 0.0004x^3 = 5.0004x^3.$$

$$\text{Therefore} \qquad |f(x)| \leq 5.0004x^3 \quad \text{whenever } x > 1.$$

This proves that $f(x)$ is $O(x^3)$. (We have used $C = 5.0004$ and $k = 1$ as our "witnesses" to this fact. We could have used $C = 6$ instead, since $|f(x)| \leq 6x^3$ whenever $x > 1$. For the purpose of checking the definition, one is as good as the other.)

...We say that $f(x)$ is $O(g(x))$ if there exist constants C and k such that

$$|f(x)| \le C|g(x)| \quad \text{whenever } x > k\,.$$

Continue with $f(x) = 5x + 0.0004x^3$: We saw that $f(x)$ is $O(x^3)$. Are the following also true for this f:

(i) $f(x)$ is $O(x)$?

(ii) $f(x)$ is $O(x^4)$?

Poll:

(A) Only (i) is true

(B) Only (ii) is true

(C) Both are true

(D) Neither is true

<u>Example:</u> Let $h(x) = -12x^5 + 5x^4 - 4x^3 + 11x$.

What can we say about $h(x)$ being $O(x^n)$ for some n?

We can use the Triangle Inequality here:

$$
\begin{aligned}
|h(x)| &= |-12x^5 + 5x^4 - 4x^3 + 11x| \\
&\leq |-12x^5| + |5x^4| + |-4x^3| + |11x| \\
&= 12|x^5| + 5|x^4| + 4|x^3| + 11|x| \\
&\leq 12|x^5| + 5|x^5| + 4|x^5| + 11|x^5| \qquad \text{for } x > 1 \\
&= 32|x^5|.
\end{aligned}
$$

Therefore $h(x)$ is $O(x^5)$.

In general, Theorem 1 says that every polynomial of degree n is $O(x^n)$.

Some useful relations (p. 223–234):

(1) x is $O(2^x)$

In fact, x^p is $O(b^x)$ for every $p > 0$ and every base $b > 1$

(2) $\log_b x$ is $O(x)$ for every base $b > 1$

(3) b^n is $O(n!)$ for every base $b > 1$

<u>Exercise:</u> Use (2) and the identity $\log_b(x^q) = q \log_b(x)$ to prove that $\log_b(x)$ is $O(\sqrt{x})$

Definition Let f and g be functions with domain $\mathbb{Z}$ or $\mathbb{R}$ (or $\mathbb{Z}^+$ or $\mathbb{R}+$) and co-domain $\mathbb{R}$.

(i) We say that $f(x)$ is $\Omega(g(x))$ if there exist constants C and k such that

$$|f(x)| \geq C|g(x)| \qquad \text{whenever } x > k\,.$$

(ii) We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is both $O(g(x))$ and $\Omega(g(x))$.

Example: We have seen that $x^4 > x^3$ whenever $x > 1$, which tells us that x^4 is $\Omega(x^3)$. (With witnesses $C = 1$ and $k = 1$)

Example: We have seen that the function $f(x) = 5x + 0.0004x^3$ is $O(x^3)$. Is $f(x)$ also $\Omega(x^3)$?

This is rather obvious: $f(x) = 5x + 0.0004x^3 \geq 0.0004x^3$ whenever $x > 0$. Therefore $f(x)$ is $\Omega(x^3)$, with witnesses $C = 0.0004$ and $k = 0$.

Hence, $5x + 0.0004x^3$ is $\Theta(x^3)$:

$$0.0004x^3 \leq 5x + 0.0004x^3 \leq 5.0004x^3 \qquad \text{whenever } x > 1.$$

Example: The Prime Number Theorem

A positive integer m is prime if $m > 1$ and the only ways to write m as the product of two positive integers are $m \times 1$ and $1 \times m$.

So the primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,...

Euclid's Theorem: The set of primes is infinite.
(Proved about 2400 years ago.)

How many primes are less than a million? Less than a trillion?

Let $\pi(n)$ be the number of primes between 1 and n.
E.g. $\pi(10) = 4$, $\pi(100) = 22$, $\pi(10^6) = 78{,}498$,
$\pi(10^{12}) \approx 4\times 10^{10}$.

The **Prime Number Theorem** (proved in 1896) describes how the function $\pi(n)$ grows as n gets large. In our terms, it implies that

$$\pi(n) \quad \text{is} \quad \Theta\left(\frac{n}{\log n}\right).$$

That is, there are constants C_1 and C_2 and k such that

$$C_1 \frac{n}{\log n} \;\leq\; \pi(n) \;\leq\; C_2 \frac{n}{\log n} \qquad \text{whenever } n > k.$$

Example: Let $f(x) = \dfrac{x^3}{\sqrt{x}+4}$ for $x > 0$.

Can we show that $f(x)$ is $\Theta(x^p)$ for some power p?
Well, $f(x) \le x^3$, so $f(x)$ is $O(x^3)$. Can we do better?

$$\frac{x^3}{\sqrt{x}+4} \le \frac{x^3}{\sqrt{x}} \quad \text{(decreasing the denominator)}$$

$$= \frac{x^3}{x^{1/2}} = x^{3-\frac{1}{2}} = x^{5/2}. \qquad \therefore\ f(x) \text{ is } O(x^{5/2}).$$

Can we also show $f(x) \ge C\,x^{5/2}$?
We would like to replace the denominator by $\sqrt{x}$ (times a constant), while increasing the denominator.
Observe that $\sqrt{x}+4 \le \sqrt{x} + 4\sqrt{x}$ whenever $x > 1$.
So whenever $x > 1$, we have $\sqrt{x}+4 \le 5\sqrt{x}$, and hence

$$\frac{x^3}{\sqrt{x}+4} \ge \frac{x^3}{5\sqrt{x}} = \frac{1}{5}\,x^{5/2}. \qquad \therefore\ f(x) \text{ is } \Omega(x^{5/2}).$$

Conclusion: $f(x)$ is $\Theta(x^{5/2})$.

Example: Let $f(n) = \sum_{j=1}^{2n} \sqrt{j}$. That is,

$f(1) = \sqrt{1} + \sqrt{2}$, $f(2) = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}$, etc.

Prove that $f(n)$ is $\Theta(n\sqrt{n})$.

(Reminder: $n\sqrt{n} = n^1 n^{\frac{1}{2}} = n^{1+\frac{1}{2}} = n^{3/2}$.)

To prove $O(n\sqrt{n})$: Notice that $f(n)$ is the sum of $2n$ terms, of which the largest one is $\sqrt{2n}$. Therefore
$f(n) \leq \sum_{j=1}^{2n} \sqrt{2n} = 2n\sqrt{2n} = (2\sqrt{2})(n\sqrt{n})$.

To prove $\Omega(n\sqrt{n})$ (this is trickier):

$$f(n) \;\geq\; \sum_{j=n+1}^{2n} \sqrt{j} \;\geq\; \sum_{j=n+1}^{2n} \sqrt{n}$$

(because: If $j \geq n+1$, then $\sqrt{j} \geq \sqrt{n+1} \geq \sqrt{n}$.) Thus

$$f(n) \;\geq\; \sum_{j=n+1}^{2n} \sqrt{n} \;=\; n \times \sqrt{n} \qquad (n \text{ terms in this sum}).$$

Since $f(n) \geq n\sqrt{n}$, we conclude that $f(n)$ is $\Omega(n\sqrt{n})$.

Since $f(n)$ is $O(n\sqrt{n})$ and $\Omega(n\sqrt{n})$, it is $\Theta(n\sqrt{n})$.

Next class: Read Section 5.1. This describes a powerful method of proof called mathematical induction. On first reading, read up to page 344.

Information about the first midterm (next Tuesday) is on eClass.

New office hours (S616 Ross and Zoom):

Mondays 2:00–3:00
Tuesdays 1:30–2:30
Fridays 1:00–2:00