EECS/MATH 1019
Section 2.5: Cardinality — Infinite Sets

February 2, 2023

Recall from Section 2.1.4:

Definition: Let $n \in \mathbb{N}$ and let S be a set. If S has exactly n elements, then we say that n is the *cardinality* of S, and we write $|S| = n$. In this case we also say that S is a *finite set*.

Examples: (a) $\{1, 3, 5, 7, 9\}$ has cardinality 5. It is a finite set.
(b) The empty set is a finite set, with cardinality 0. (Recall $0 \in \mathbb{N}$.)
...
(d) If B is a finite set, and $A \subseteq B$, then A is finite and
$|B - A| = |B| - |A|$.
In particular, if $A \subseteq B$, then $|A| \leq |B|$.

An *infinite set* is a set that is not finite. E.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Today we shall focus on infinite sets, with two principle themes:

(1) Some infinite sets are "bigger" than others (but not in the way you might expect)

(2) There are implications in theoretical computer science: some things cannot be computed

First let's have another look at finite sets.

Let $n \in \mathbb{Z}^+$. A set S has cardinality n (i.e. $|S| = n$) if and only if there is a bijection from $\{1, 2, \ldots, n\}$ to S.

(Recall *bijection* is also called *one-to-one correspondence*)

Example (a) Let GL be the set of Great Lakes: $GL = \{$Erie, Huron, Michigan, Ontario, Superior$\}$. Then $|GL| = 5$, and here is a bijection $f : \{1, 2, 3, 4, 5\} \to GL$: $f(1) =$ Ontario, $f(2) =$Erie, $f(3) =$Huron, $f(4) =$Michigan, $f(5) =$Superior.

Property A: Two nonempty finite sets A and B have the same cardinality if and only if there is a bijection from A to B.

E.g. $\{$Medicine, Physics, Chemistry, Peace, Literature$\}$ has cardinality 5, so there is a bijection from this set to GL.

Big Idea: We also use Property A for infinite sets. In fact, that will be how we define cardinality of infinite sets.

Definition 1: Two nonempty sets A and B have the same cardinality if and only if there is a bijection from A to B. We write $|A| = |B|$ to denote this situation.

Examples: Consider the sets $\mathbb{Z}$ (all integers), $\mathbb{Z}^+ = \{1, 2, 3 \ldots\}$, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and $Even = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

First consider $\mathbb{Z}^+$ and $\mathbb{N}$. Note that $\mathbb{Z}^+ \subset \mathbb{N}$ but $|\mathbb{Z}^+| = |\mathbb{N}|$ because there is a bijection $g : \mathbb{Z}^+ \to \mathbb{N}$ defined by $g(x) = x - 1$.

Next consider $\mathbb{Z}$ and $Even$. We have a bijection $h : \mathbb{Z} \to Even$ defined by $h(x) = 2x$, so $|\mathbb{Z}| = |Even|$.

Next consider $\mathbb{N}$ and $\mathbb{Z}$.

$$\begin{array}{cccccccccc} \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{Z} & 0 & \text{-1} & 1 & \text{-2} & 2 & \text{-3} & 3 & \cdots \end{array}$$

This is a bijection! (Exercise: Show that we can write this bijection as $k(x) = (-1)^x \lceil x/2 \rceil$.) We conclude that $|\mathbb{N}| = |\mathbb{Z}|$.

Therefore $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, and $Even$ all have the same cardinality. (This is because if $f_1 : A \to B$ and $f_2 : B \to C$ are bijections, then $f_2 \circ f_1 : A \to C$ is also a bijection. Exercise: Prove this.)

Definition 3: A set is said to be countable if it is finite or if it has the same cardinality as $\mathbb{Z}^+$.
An uncountable set is a set that is not countable.

We have seen that $\mathbb{N}$, $\mathbb{Z}$ and *Even* are all countable. The set $\mathbb{Q}$ of rational numbers is also countable (Example 4, page 182).

However, the set $\mathbb{R}$ of all real numbers is not countable. This was proved in 1879 by Georg Cantor. At the time, it was a big surprise that not all "infinite numbers" are the same!
See Example 5 (p. 183) for the proof. We will do a similar proof in class later today in a different context.

Remark: Not all uncountable sets have the same cardinality as each other. In particular, there is an uncountable set U such that $|U| \neq |\mathbb{R}|$. (An example is the power set $\mathcal{P}(\mathbb{R})$, the set of all subsets of $\mathbb{R}$.)

Observation: A set is countable if and only if all of its elements can be listed as the terms of a sequence $a_0, a_1, a_2, \ldots$

Theorem 1: The union of two countable sets is a countable set.

Why is this true? Let A and B be two countable sets. Then we can write $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$.

$$\begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \ldots \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \ldots \end{array}$$

How can we make one sequence out of both rows? Here is one way:

$$a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a_4,\ b_4,\ \ldots$$

(If B is finite, say, then just use a's when you use up all the b's. If some elements are in both A and B, say $a_9 = b_5$, then skip over any element you come to that has already appeared in the list.)

Similarly, the union of three countable sets is countable:

$$\begin{array}{cccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \ldots \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \ldots \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \ldots \end{array}$$

Sequence: $a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2,\ a_3,\ \ldots$

What about the union of a countable number of finite sets?
Assume that $A_1, A_2, A_3, A_4, \ldots$ are finite sets, and let

$$W \;=\; \bigcup_{n=1}^{\infty} A_n \;=\; A_1 \cup A_2 \cup A_3 \cup \ldots$$

Then we can make a sequence that contains all elements of W:
First list all of the elements of A_1, then list all the elements of A_2, then list all the elements of A_3, and so on.
This tells us that W is countable. This proves

Theorem 1.A: The union of a countable number of finite sets is a countable set.

Example: $\mathbb{N}^2$: Recall that

$$\mathbb{N}^2 \;=\; \mathbb{N} \times \mathbb{N} \;=\; \{(j,k) \,|\, j \in \mathbb{N} \text{ and } k \in \mathbb{N}\} \;.$$

We shall use the above ideas to prove that $\mathbb{N}^2$ is a countable set.

Theorem 1.A: The union of a countable number of finite sets is a countable set.

$$\begin{aligned}
\mathbb{N}^2 &= \mathbb{N} \times \mathbb{N} = \{(j,k) \mid j \in \mathbb{N} \text{ and } k \in \mathbb{N}\} \\
&= \{ (0,0), (0,1), (0,2), (0,3), \ldots \\
&\quad (1,0), (1,1), (1,2), (1,3), \ldots \\
&\quad (2,0), (2,1), (2,2), (2,3), \ldots \\
&\quad (3,0), (3,1), (3,2), (3,3), \ldots \} \\
&= \{ (0,0) \} \cup \{ (0,1), (1,0) \} \cup \{ (0,2), (1,1), (2,0) \} \\
&\quad \cup \{ (0,3), (1,2) (2,1), (3,0) \} \cup \cdots
\end{aligned}$$

Now, for each $n \in \mathbb{N}$, let $A_n = \{(j,k) \in \mathbb{N}^2 \mid j + k = n\}$, so that

$A_0 = \{(0,0)\}$, $A_1 = \{ (0,1), (1,0) \}$, $A_2 = \{ (0,2), (1,1), (2,0) \}$

and so on. Note that each set A_n is finite. Then

$$\mathbb{N}^2 = \bigcup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup A_2 \cup A_3 \cup \ldots$$

Therefore, by Theorem 1.A, $\mathbb{N}^2$ is a countable set.

What can we say about the cardinalities of finite sets A and B if there is an injective (i.e. one-to-one) function from A to B? Then A has smaller cardinality than B, i.e. $|A| \leq |B|$.

The same thing holds for infinite sets. In fact, the existence of an injective function from A to B is the definition of the assertion $|A| \leq |B|$ (Definition 2).

The following result is obvious for finite sets, but harder to prove for infinite sets:

Theorem 2: If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

For examples of Theorem 2 in use, see the text or the end of today's slides.

Uncomputable Functions (p. 185)

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be computable if there is a computer program in some programming language (or, an algorithm) that can evaluate $f(n)$ for every natural number n. A function that is not computable is said to be uncomputable.

E.g. The factorial function $n!$ is computable.

So is "the largest prime number that is a factor of n".

So is $F(n) =$ "the number of ordered pairs $(j,k) \in \mathbb{N}^2$ such that $3j^4 + 5k^6 = n^7$."

So are many other functions that are easily written down.

Question: Are there any uncomputable functions from $\mathbb{N}$ to $\mathbb{N}$?
Answer: Yes!

A proof of this answer is based on two facts:
(1) The number of possible computer programs is countable (because every program can be expressed as a finite sequence of characters from a finite "alphabet"; apply Theorem 1.A) (Ex. 37);
(2) The number of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

We shall prove that the number of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

The proof will be by contradiction.

Assume that the number of functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.

Then we can put all of these functions in a sequence $f_0, f_1, f_2, f_3, \ldots$ where each f_n is a function from $\mathbb{N}$ to $\mathbb{N}$.

$$\begin{array}{lllllll}
f_0: & f_0(0) & f_0(1) & f_0(2) & f_0(3) & f_0(4) & \ldots \\
f_1: & f_1(0) & f_1(1) & f_1(2) & f_1(3) & f_1(4) & \ldots \\
f_2: & f_2(0) & f_2(1) & f_2(2) & f_2(3) & f_2(4) & \ldots \\
f_3: & f_3(0) & f_3(1) & f_3(2) & f_3(3) & f_3(4) & \ldots \\
\vdots & & & & & &
\end{array}$$

For example, if the first few functions are $f_0(n) = n^2$, $f_1(n) = 3n$, $f_2(n) = n!$, $f_3(n) = 2n + 1$, then we would have

$$\begin{array}{lcccccc}
f_0: & 0 & 1 & 4 & 9 & 16 & \ldots \\
f_1: & 0 & 3 & 6 & 9 & 12 & \ldots \\
f_2: & 1 & 1 & 2 & 6 & 24 & \ldots \\
f_3: & 1 & 3 & 5 & 7 & 9 & \ldots
\end{array}$$

Can we find a function that is different from every function in this infinite list? The assumption says NO.

We now define another function $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$g(n) = f_n(n) + 10 \qquad \text{for every } n \in \mathbb{N}.$$

Here is how to think of g for our example:

f_0 :	0	1	4	9	16	...
f_1 :	0	3	6	9	12	...
f_2 :	1	1	2	6	24	...
f_3 :	1	3	5	7	9	...
$\vdots$						

The entries $f_0(0)$, $f_1(1)$, $f_2(2)$, $f_3(3), \ldots$ are in red

So we obtain

$g(0) = 10$, $g(1) = 13$, $g(2) = 12$, $g(3) = 17, \ldots$

Whatever the actual functions $f_0, f_1, f_2, \ldots$ are, we see

g is not f_0 because $g(0) \neq f_0(0)$ (in fact, $g(0) = f_0(0) + 10$)

g is not f_1 because $g(1) \neq f_1(1)$ (in fact, $g(1) = f_1(1) + 10$)

$\vdots$

g is not f_{79} because $g(79) \neq f_{79}(79)$ $\quad (g(79) = f_{79}(79) + 10)$

$\vdots$

So for every natural number n, the function g is not f_n because $g(n) \neq f_n(n)$ (in fact, $g(n) = f_n(n) + 10$)

$$g(n) \;=\; f_n(n) + 10 \qquad \text{for every } n \in \mathbb{N}\,.$$

$$\begin{array}{llllllll}
f_0: & f_0(0) & f_0(1) & f_0(2) & f_0(3) & f_0(4) & \dots \\
f_1: & f_1(0) & f_1(1) & f_1(2) & f_1(3) & f_1(4) & \dots \\
f_2: & f_2(0) & f_2(1) & f_2(2) & f_2(3) & f_2(4) & \dots \\
f_3: & f_3(0) & f_3(1) & f_3(2) & f_3(3) & f_3(4) & \dots \\
\vdots & & & & & \ddots &
\end{array}$$

So g is not in the list $f_0, f_1, f_2, \ldots$.
But we assumed that this list contains ALL functions from $\mathbb{N}$ to $\mathbb{N}$.

CONTRADICTION!

We conclude that there uncountably many functions from $\mathbb{N}$ to $\mathbb{N}$.
This completes the proof of fact (2).

And fact (1) says that only countably many of these functions are computable.

Therefore there exist (many!) uncomputable functions.

Additional examples

Example: We shall use Theorem 1.A to prove that $\mathbb{Q}$, the set of rational numbers, is a countable set.

We can express the set of rational numbers as

$$\mathbb{Q} \ = \ \left\{ \frac{m}{n} \ \middle| \ m \in \mathbb{Z}, \ n \in \mathbb{Z}, \ \text{ and } n > 0 \right\}$$

For each positive integer k, let

$$A_k \ = \ \left\{ \frac{m}{n} \ \middle| \ m \in \mathbb{Z}, \ n \in \mathbb{Z}, \ n > 0, \ \text{ and } |m| + |n| \ = \ k \right\} .$$

That is,

$$A_1 = \left\{ \frac{0}{1} \right\}, \ \ A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\}, \ \ A_3 = \left\{ \frac{2}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{-2}{1} \right\}$$

and so on. Since every rational number is in A_k for some k, it follows that

$$\mathbb{Q} \ = \ \bigcup_{k=1}^{\infty} A_k \, .$$

Also, for every k, we see that A_k is a finite set. Therefore, Theorem 1.A tells us that $\mathbb{Q}$ is countable.

Example: Here is another way to prove that $\mathbb{N}^2$ is countable.
Recall

Theorem 2: If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

We shall use Theorem 2 to prove that $|\mathbb{N}| = |\mathbb{N}^2|$.
(That is, we shall prove that the set of natural numbers has the same cardinality as the set of ordered pairs of natural numbers.)

To show $|\mathbb{N}| \leq |\mathbb{N}^2|$: It is easy to find a one-to-one function from $\mathbb{N}$ to $\mathbb{N}^2$. E.g. $f(x) = (x, 0)$.

To finish the proof, we need to show $|\mathbb{N}^2| \leq |\mathbb{N}|$.
That is, we need to find a one-to-one function from $\mathbb{N}^2$ to $\mathbb{N}$.
(This is the trickier part!)

Here's one way to construct a one-to-one function H from $\mathbb{N}^2$ to $\mathbb{N}$.

Let $(j, k) \in \mathbb{N}^2$. Then j and k are natural numbers.
Let $H(j, k)$ be the number represented (in base 10) by a string of j 2's followed by a string of k 5's.

E.g. $H(3, 4) = 2225555$, $\quad H(12, 1) = 2222222222225$,
$H(6, 0) = 222222$, $\quad H(0, 6) = 555555$,
and we'll specially define $H(0, 0)$ to be 0.

Clearly "most" natural numbers are not in the range of H. But any number x that is in the range of H obviously has only one possible preimage (counting the number of 2's gives j, and count 5's gives k).

Thus $H : \mathbb{N}^2 \to \mathbb{N}$ is one-to-one.

This proves that $|\mathbb{N}^2| \leq |\mathbb{N}|$.
This completes the proof that $|\mathbb{N}^2| = |\mathbb{N}|$.

Remark: Without Theorem 2, it would not yet be clear whether there exists a bijection from $\mathbb{N}^2$ to $\mathbb{N}$. Theorem 2 assures us that such a bijection exists, although we don't know what it is!

Next class: Read Section 3.2. This section is motivated as a tool for describing the efficiency of algorithms (although we will not have much to say about algorithms in this course).

Information about the first midterm, on Tuesday February 14, is on eClass. Two sample tests are also posted.

Homework updates:

• Problem Set A: Two proofs to write up and submit to Crowdmark by Thursday February 2 (tonight!). The questions are posted on our eClass page.

• Homework assignment 3 (in Connect) is due Sunday February 5.