EECS/MATH 1019
Section 2.3: Functions
Section 2.4: Sequences and Sums

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Section 2.3: Functions

We write $f : A \to B$ to say that f is a function from the set A to the set B.
That is, f assigns one element of B to each element of A.
For $a \in A$, we write $f(a)$ to denote the unique element of B that is assigned to a.

<u>Examples:</u>

(a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.
E.g. $g(3) = 9$, $g(8) = 64 = g(-8)$.

(b) Let *Prov* be the set of Canadian provinces, and let *Cit* be the set of Canadian cities.
We define the function $C : Prov \to Cit$ by the rule that $C(x)$ is the capital city of province x.

E.g. C(Ontario) is Toronto; C(Alberta) is Edmonton.

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.
(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

We can also represent a function $f : A \to B$ by a subset of $A \times B$ consisting of all ordered pairs of the form $(a, f(a))$.

(b) For C, this subset includes (ON,Toronto), (AB, Edmonton), (QC, Quebec City), and seven other ordered pairs (one for each province).

(a) For g, this subset includes $(0, 0)$, $(3, 9)$, $(-3, 9)$, $(\sqrt{2}, 2)$, and many other points. When viewed as a subset of the plane $\mathbb{R}^2$ with (x, y) coordinate axes, this set is called the *graph* of the function (Sec. 2.3.4).

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

Images: For the function $f : A \to B$:

For $a \in A$, let $b = f(a)$. Then b is the image of a, and a is a preimage of b.

E.g. (a) For $g(x) = x^2$: The number 9 is the image of 3, and 3 is a preimage of 9. Observe that -3 is also a preimage of 9. (This is why we say "a preimage" instead of "the preimage.")

(b) For $C : Prov \to Cit$, the image of Manitoba is Winnipeg. A preimage of Toronto is Ontario. In this case, no city has more than one preimage. Some cities have no preimage (e.g. Hamilton, since Hamilton is not the capital of any province).

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

Terminology: For $f : A \to B$, we say that A is the domain and B is the codomain of f.

(a) For the function g, the domain is $\mathbb{R}$ and the codomain is $\mathbb{R}$.

(b) For the function C, the domain is $Prov$ and codomain is Cit.

The range (or image) of f is the set of all the images of elements of A.

(a) The range of g is the set of nonnegative real numbers.

(b) The range of C is the set of the ten provincial capitals.

Note the difference between codomain and range.

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

Let $S \subseteq A$. The image of S under f is the set of all images of elements of S.

(b) The image of { New Brunswick, Nova Scotia, PEI} under C is the set of the three corresponding capitals: {Fredericton, Halifax, Charlottetown}.

(a) **Poll:** The image of $[-2, 2]$ under g is

(A) $[-4, 4]$

(B) $[0, 4]$

(C) $\{4\}$

(D) $\{0, 1, 4\}$

Answer: (B)

Notation: The image of the set S under f is written $f(S)$. It equals $\{f(x) \mid x \in S\}$.

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

A function $f : A \to B$ is one-to-one, or injective, if and only if no element of B has more than one preimage.

This is equivalent to saying

$\forall u \in A, \forall w \in A\,([f(u) = f(w)] \to [u = w])$.

(a) Is the function g injective?

No, because 9 has two preimages, namely -3 and $+3$.

Alternative answer: No, because $g(-3) = g(3)$ but $-3 \neq 3$.

(b) Is the function C injective?

Yes. No city is the capital of two different provinces.

Examples: (a) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

(b) $C : Prov \to Cit$ defined by $C(x) =$ the capital of x.

A function $f : A \to B$ is onto, or surjective, if and only if every element of B has at least one preimage.

This is equivalent to saying $\forall b \in B, \exists a \in A\, (f(a) = b)$.

This is also equivalent to saying that the range equals the codomain.

Poll: Which of the above functions g and C are surjective?

(A) both
(B) neither
(C) only g
(D) only C

Answer: Neither.

The function g is not surjective because -1 is in the codomain but has no preimage, i.e. there is no number x such that $x^2 = -1$.

The function C is not surjective because Mississauga is in *Cit* but has no preimage. That is, there is no province whose capital is Mississauga.

A function $f : A \to B$ is one-to-one, or injective, if and only if no element of B has more than one preimage.

A function $f : A \to B$ is onto, or surjective, if and only if every element of B has at least one preimage.

Subtler examples: Which of these functions are one-to-one? onto?

$g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

$h : \mathbb{R} \to [0, \infty)$ defined by $h(x) = x^2$.

$k : [0, \infty) \to [0, \infty)$ defined by $k(x) = x^2$.

The function g is NOT one-to-one because e.g. $g(-2) = 4 = g(2)$, and it is NOT onto because -1 is in the codomain but has no preimage, i.e. there is no number x such that $x^2 = -1$.

The function h is NOT one-to-one because e.g. $h(-2) = 4 = h(2)$, and it IS onto because for every b in $[0, \infty)$, there is an $a \in \mathbb{R}$ such that $a^2 = b$ (a could be $\sqrt{b}$, as well as $-\sqrt{b}$).

The function k IS one-to-one AND onto. This is because for every b in $[0, \infty)$, there is a UNIQUE $a \in [0, \infty)$ such that $a^2 = b$, namely $\sqrt{b}$ (the *positive square root of* b).

Example (c) Let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = 3x + 5$.

Show that L is injective (one-to-one) and surjective (onto).

Injective: Assume $L(u) = L(w)$. Then

$$\begin{aligned} 3u + 5 &= 3w + 5 && \text{Subtract 5 from both sides:} \\ 3u &= 3w && \text{Divide both sides by 3:} \\ u &= w. \end{aligned}$$

This proves that $L(u) = L(w) \to u = w$. Therefore L is injective.

Surjective: Let $b \in \mathbb{R}$. Must there be an x such that $L(x) = b$?

$$\begin{aligned} L(x) = b \quad &\leftrightarrow \quad 3x + 5 = b \\ &\leftrightarrow \quad 3x = b - 5 \\ &\leftrightarrow \quad x = \frac{b-5}{3}. \end{aligned}$$

Indeed, $L\left(\frac{b-5}{3}\right) = 3\left(\frac{b-5}{3}\right) + 5 = b.$ So, YES.

This proves that L is surjective.

A function that is injective and surjective is said to be bijective. We also say that such a function is a bijection, or a one-to-one correspondence.

Example (c) Recall $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = 3x + 5$. We proved that L is injective and surjective. Therefore L is *bijective*.

Example (a1) Recall $k : [0, \infty) \to [0, \infty)$ defined by $k(x) = x^2$. We saw that k is bijective.

Example (b1) Let Cap be the set of Canadian provincial capital cities. Define a new function $CC : Prov \to Cap$ by $CC(x) =$ the capital of x. Then CC is bijective (unlike $C : Prov \to Cit$.)

Every bijection $f : A \to B$ has an *inverse function* $f^{-1} : B \to A$, which we shall define on the next slide.

In Example (a1), $k^{-1}(b) = \sqrt{b}$.

In Example (b1), $CC^{-1} : Cap \to Prov$. E.g. $CC^{-1}(\text{Toronto}) =$ Ontario, and $CC^{-1}(\text{Victoria}) =$ British Columbia.

Assume $f : A \to B$ is bijective. The *inverse function* of f, denoted f^{-1}, is the function $f^{-1} : B \to A$ such that the image of $b \in B$ under f^{-1} equals the preimage of b under f.

Example (d): Define $M : \mathbb{R} \to \mathbb{R}$ by $M(x) = x^3$.
Then M is a bijection and $M^{-1}(b) = b^{1/3}$.

Example (c): Let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = 3x + 5$.

RECALL: Let $b \in \mathbb{R}$. Must there be an x such that $L(x) = b$?

$$L(x) = b \;\leftrightarrow\; 3x + 5 = b \;\leftrightarrow\; 3x = b - 5$$
$$\leftrightarrow\; x = \frac{b-5}{3}.$$

Indeed, $L\left(\frac{b-5}{3}\right) = 3\left(\frac{b-5}{3}\right) + 5 = b.$ So, YES.

This calculation shows that $L^{-1}(b) = \frac{b-5}{3}$. We also see

$$L\left(L^{-1}(b)\right) = b \quad \text{and} \quad L^{-1}(L(x)) = \frac{(3x+5) - 5}{3} = x.$$

The last relations hold more generally:

If the inverse function of $f : A \to B$ is $f^{-1} : B \to A$, then

$$\begin{aligned} f^{-1}(f(a)) &= a \qquad \text{for every } a \text{ in } A, \quad \text{and} \\ f\left(f^{-1}(b)\right) &= b \qquad \text{for every } b \text{ in } B. \end{aligned}$$

Recall these functions:

$h : \mathbb{R} \to [0, \infty)$ defined by $h(x) = x^2$.

$k : [0, \infty) \to [0, \infty)$ defined by $k(x) = x^2$.

We saw that h is not a bijection, hence it has no inverse function.
The function k is a bijection, and its inverse is $k^{-1}(b) = \sqrt{b}$.

$$\text{For } a \in [0,\infty), \qquad k^{-1}(k(a)) = \sqrt{a^2} = a \quad (\text{since } a \geq 0),$$

$$\text{For } b \in [0,\infty), \qquad k(k^{-1}(b)) = \left(\sqrt{b}\right)^2 = b.$$

Could $k^{-1}(b) = \sqrt{b}$ also be an inverse function of h?

$$\text{For } b \in [0,\infty), \qquad h(k^{-1}(b)) = \left(\sqrt{b}\right)^2 = b.$$

BUT: For $a = -3 \in \mathbb{R}$, $\qquad k^{-1}(h(-3)) = \sqrt{(-3)^2} = +3 \neq -3$.

So, NO.

One more piece of notation:

Assume $g : A \to B$ and $h : B \to C$. Then for every $a \in A$, $h(g(a))$ is in C (because $g(a) \in B$).
So $h(g(a))$ defines a function from A to C, called the composition of h and g, and written $h \circ g$. That is, $h \circ g : A \to C$.

Notice that $g \circ h$ does not necessarily make sense in the above situation:
We have $h(x) \in C$ for $x \in B$, so $g(h(x))$ may not be defined!

A few more things for you to read about in Section 2.3:

Subsection 2.3.5:

Floor function $\lfloor x \rfloor$ and ceiling function $\lceil x \rceil$ for $x \in \mathbb{R}$, with range $\mathbb{Z}$.

E.g. $\lfloor 5.13 \rfloor = 5$ ("round down") and $\lceil 5.13 \rceil = 6$ ("round up")

Factorial function $n! = 1 \times 2 \times 3 \times \cdots \times n$ for $n \in \mathbb{N}$, with codomain $\mathbb{Z}^+$

Subsection 2.3.6: Partial functions: May not be defined for every element of the domain(!).
E.g. the partial function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$
Don't worry about this unless you see the phrase "partial function" stated explicitly

Section 2.4: Sequences and Summations

A *sequence* is an ordered list of things. Unless we say "finite sequence," we assume that the sequence is infinitely long.
Here are some examples of infinite sequences of numbers:

(1) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(2) 1, 2, 4, 8, 16,...

(3) 2, 0, 2, 0, 2,...

(4) 3, 3.1, 3.14, 3.141, 3.1415, ...

Notation: $a_1, a_2, a_3, \ldots$; or $\{a_n\}$; or $a_0, a_1, a_2, \ldots$ [or use some other letter]
For (1): $a_n = \frac{1}{n}$ for $n \geq 1$.
There are various ways to describe (2):
- $b_n = 2^{n-1}$ for $n \geq 1$
- $c_n = 2^n$ for $n \geq 0$
- $b_1 = 1$ and $b_n = 2\, b_{n-1}$ for $n \geq 2$ [a *recurrence relation*]

Sequence (2) is an example of a *geometric progression*.

For the third example: 2, 0, 2, 0, 2,... can be described as follows:

- $d_k = \begin{cases} 2 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$ or
- $d_k = 1 - (-1)^k$ for $k \geq 1$ or
- $d_1 = 2$ and $d_k = 2 - d_{k-1}$ for $k \geq 2$

For the fourth example, 3, 3.1, 3.14, 3.141, 3.1415, ... :
The n^{th} term in the sequence is π truncated to n digits.

We can also view a sequence $a_1, a_2, a_3, \ldots$ as a function with domain $\{1, 2, 3, \ldots\}$, writing a_n instead of $f(n)$.

A finite sequence (e.g. the digits of your student number, or the names of the people in this class in alphabetical order) is also called a string. This section mainly discusses infinite sequences.

Example (5): The sequence of factorials, defined by

$$
\begin{aligned}
1! &= 1,\\
2! &= 2 \times 1 = 2\\
3! &= 3 \times 2 \times 1 = 6\\
4! &= 4 \times 3 \times 2 \times 1 = 24\\
&\vdots\\
n! &= n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1
\end{aligned}
$$

and

$0! = 1.$

Observe that factorials satisfy the relations

$$4! = 4 \times [3!], \quad 3! = 3 \times [2!], \quad 2! = 2 \times [1!], \quad 1! = 1 = 1 \times [0!]$$

and in general

$$n! \;=\; n \times [(n-1) \times (n-2) \times \cdots \times 2 \times 1] \;=\; n \times [(n-1)!] \quad \text{for } n \geq 1.$$

So if we define a sequence k_n by the recurrence relation

$$k_0 = 1 \quad \text{and} \quad k_n = n \times k_{n-1} \quad \text{for } n \geq 1,$$

then $k_n = n!$ for every $n \in \mathbb{N}$.

Example (6): 3, 13, 23, 33, 43,...

We can write this sequence as $a_1, a_2, a_3, \ldots$ with the "closed formula" $a_n = 3 + 10(n - 1)$.

This is an example of an *arithmetic progression* with *initial term* 3 and *common difference* 10.

Subsections 2.4.3–2.4.4 give more examples of sequences, notably the *Fibonacci sequence* in Definition 5.
We will revisit sequences and recurrence relations in much more detail in Chapters 5 and 8.

Section 2.4.4: Summation

Notation: For a real sequence $\{a_n\}$, we write

$$a_4 + a_5 + a_6 + a_7 + a_8 \;=\; \sum_{n=4}^{8} a_n \;=\; \sum_{w=4}^{8} a_w$$

and more generally

$$a_m \;+\; a_{m+1} \;+\; \cdots \;+\; a_n \;=\; \sum_{i=m}^{n} a_i \,.$$

Example: The sum of the first 6 terms of the sequence 1, 2, 4, 8, 16,. . . is

$$1+2+4+8+16+32 \;=\; \sum_{j=0}^{5} 2^j \;=\; \sum_{k=1}^{6} 2^{k-1} \;=\; 63.$$

See Theorem 1 for a formula for the sum of terms in a geometric sequence (and as an exercise, check that it works in this case).

Notice that Section 2.2.3 introduced similar notation for unions and intersections:

$$J_3 \cup J_4 \cup J_5 \cup J_6 \;=\; \bigcup_{n=3}^{6} J_n \qquad\qquad J_3 \cap J_4 \cap J_5 \cap J_6 \;=\; \bigcap_{n=3}^{6} J_n$$

For our example from the class of January 26:
Suppose $J_n \;=\; \{n, n+1, n+2\}$ for each n.
Then

$$\bigcup_{n=3}^{6} J_n \;=\; \{3,4,5\} \cup \{4,5,6\} \cup \{5,6,7\} \cup \{6,7,8\} \;=\; \{3,4,5,6,7,8\}$$

and

$$\bigcap_{n=3}^{6} J_n \;=\; \{3,4,5\} \cap \{4,5,6\} \cap \{5,6,7\} \cap \{6,7,8\} \;=\; \emptyset\,.$$

Example: What is the sum of the first 100 positive integers? That is,

$$1+2+3+4+\cdots+99+100 = \sum_{k=1}^{100} k = ?$$

There is a story about Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians ever. When he was 8 years old, his teacher asked the class to add up the numbers from 1 to 100. Gauss noticed

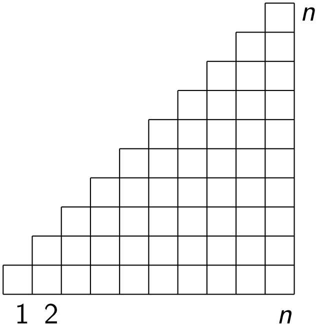
$$\begin{aligned} 1+100 &= 101 \\ 2\ +\ 99 &= 101 \\ 3\ +\ 98 &= 101 \\ &\ \vdots \\ 50+51 &= 101 \end{aligned}$$

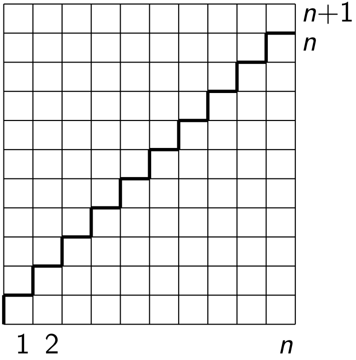
and quickly concluded that the sum of the numbers from 1 to 100 must be $101 \times 50 = 5050$.

The formula for the sum of the first n positive integers is

$$1 + 2 + 3 + \cdots + n \;=\; \sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}\,.$$

This can be proven in many ways. Gauss' approach is one way. There is also a visual argument:





The number of squares in the left picture is $1 + 2 + \cdots + n$.
This equals half of the number of squares in the right picture.
The right picture is a rectangle with $n \times (n+1)$ little squares.

Other special formulas for sums are given in Table 2 (p. 176). These are only for reference; you don't need to memorize this table.

Sums of infinitely many terms (also called "infinite series") are difficult to deal with. They are usually treated in a Calculus II course. They lead to some very interesting mathematics, such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We will not discuss infinite sums in this course.

Example (Double summation):

$$\begin{aligned}
\sum_{i=2}^{4}\sum_{j=3}^{5}(i+j)^2 &= \sum_{i=2}^{4}\left(\sum_{j=3}^{5}(i+j)^2\right) \\
&= \sum_{j=3}^{5}(2+j)^2 + \sum_{j=3}^{5}(3+j)^2 + \sum_{j=3}^{5}(4+j)^2 \\
&= \left((2+3)^2+(2+4)^2+(2+5)^2\right) \\
&\quad + \left((3+3)^2+(3+4)^2+(3+5)^2\right) \\
&\quad\quad + \left((4+3)^2+(4+4)^2+(4+5)^2\right)
\end{aligned}$$

which turns out to be 453.

Next class: Read Section 2.5. It is mostly about infinity, and is not easy to grasp on first reading.

The next homework is posted:
• Problem Set A: Two proofs to write up and submit online through Crowdmark, due Thursday February. It is posted on our eClass page.
• Homework assignment 3: Questions in Connect, as usual, due Sunday February 5.

Information about the first midterm, on Tuesday February 14, will be posted on eClass soon. It will cover everything up to and including Section 2.5.