EECS/MATH 1019
Sections 2.1 and 2.2:
Sets — Introduction/Review

January 26, 2023

A *set* is an unordered collection of distinct objects.
The objects are the *elements* of the set.

<u>Notation:</u> We write $x \in A$ to mean that x is an element of A, and $x \notin A$ to mean that x is not an element of A.

E.g. Consider the set $A = \{1, 3, 5, 7, 9\}$. Then $3 \in A$ and $4 \notin A$.

E.g. We often write $\mathbb{N}$ to denote the set of nonnegative integers $\{0, 1, 2, 3, 4, \ldots\}$. The text calls this the set of *natural numbers*.

We can also write the above set A as

$$A = \{x \in \mathbb{N} \,|\, x \text{ is odd and } x < 11\}\,.$$

Our set $A = \{1, 3, 5, 7, 9\}$ is the same as the set
$B = \{5, 1, 3, 9, 7\}$.
That is, $A = B$. This is because the sets A and B have exactly the same elements as each other:

$$\forall x\,[\,(x \in \{1, 3, 5, 7, 9\}) \leftrightarrow (x \in \{5, 1, 3, 9, 7\})\,].$$

Also note that "repetitions don't count" in a set: the set $\{1, 3, 1, 5, 1, 7, 7, 9\}$ should be written as $\{1, 3, 5, 7, 9\}$.

Let C and D be two sets. We say that C is a subset of D if every element of C is an element of D. We write this as $C \subseteq D$. E.g.

$$\{1, 3, 9\} \subseteq \{1, 3, 5, 7, 9\} \quad \{1, 3, 5, 7, 9\} \subseteq \mathbb{N} \quad \{3\} \subseteq \{1, 3, 9\}$$

Which of the following are correct?

(a)	$\{3\} \subseteq \{1, 3, 9\}$	(b)	$\{3\} \in \{1, 3, 9\}$
(c)	$3 \subseteq \{1, 3, 9\}$	(d)	$3 \in \{1, 3, 9\}$

More terminology:

If C is a subset of D, then D is a superset of C. We write this as $D \supseteq C$.

Thus $D \supseteq C$ if and only if $C \subseteq D$.

The notation "$E \not\subseteq F$" means "E is <u>not</u> a subset of F."

Every set is a subset of itself: e.g. $\{1, 3, 9\} \subseteq \{1, 3, 9\}$. Sometimes we only want to consider subsets of a set G that are different from G. We call these *proper* subsets. That is:

We say that H is a proper subset of G if $H \subseteq G$ and $H \neq G$. We write this as $H \subset G$.
(Note the analogy to $\leq$ and $<$.)

Again, let $A = \{1, 3, 5, 7, 9\}$.

What is $\{x \in A \,|\, x \text{ is even}\}$? Answer: The set with no elements. This is called the *empty set*, and we write it $\emptyset$ or $\{\}$.

Observe that $\emptyset \subseteq \{1, 3, 5, 7, 9\}$. In fact, the empty set is a subset of every set!

Question: What is $\{\emptyset\}$? It is not the empty set. Rather, it is the set whose single element is the empty set. The empty set is a mathematical object, so it can be the element of a set.

For example, the *power set* of a set S is the set of all subsets of S (Sec. 2.1.5). Thus, the power set of $\{1, 3, 9\}$ is

$$\{\emptyset, \{1\}, \{3\}, \{9\}, \{1, 3\}, \{1, 9\}, \{3, 9\}, \{1, 3, 9\}\}$$

(this is a set with 8 elements, each of which is a set).

Some special sets of numbers:

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers (from the German word “Zahlen” meaning “numbers”)

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the set of positive integers (note: some books refer to this set as $\mathbb{N}$, and say that 0 is not a natural number)

$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$, the set of rational numbers (from the word “quotient”)

$\mathbb{R}$, the set of real numbers

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, the set of positive real numbers

Intervals: For example,

$[1.32\,, 7.04] = \{x \in \mathbb{R} \mid 1.32 \leq x \leq 7.04\}$,
$(1.32\,, 7.04) = \{x \in \mathbb{R} \mid 1.32 < x < 7.04\}$,
$[1.32, 7.04) = \{x \in \mathbb{R} \mid 1.32 \leq x < 7.04\}$,
$(1.32, 7.04] = \{x \in \mathbb{R} \mid 1.32 < x \leq 7.04\}$,

More about the "set builder" notation (Sec. 2.1.8)

Here are two examples of sets we have mentioned using the "set builder" notation:

$$\begin{aligned} [1.3, 7] &= \{x \in \mathbb{R} \,|\, 1.3 \le x \le 7\} \\ \{1, 3, 5, 7, 9\} &= \{x \in \mathbb{N} \,|\, x \text{ is odd and } x < 11\} \end{aligned}$$

Both of these are of the form

$$S = \{x \in U \,|\, P(x)\}$$

where P is a predicate (as in Section 1.4) (i.e. $P(x)$ is a proposition for each x), and U is the domain of the predicate.

For $[1.3, 7]$, U is $\mathbb{R}$ and $P(x)$ is "$1.3 \le x \le 7$."
For $\{1, 3, 5, 7, 9\}$, U is $\mathbb{N}$ and $P(x)$ is "x is odd and $x < 11$."

We interpret $\{x \in U \,|\, P(x)\}$ as "the set of all x in U such that $P(x)$ is true." This is also called the *truth set* of P (in U).

In the context of sets, we often call U the "universal set." (Informally, it is the set of all objects currently under consideration.)

More notation for quantifiers:

In the class of Jan 12, to emphasize the importance of being clear about the domain of a quantifier, we considered the statement

$$\exists x\,(5 < x^2 < 8)\,.$$

We noted that it is False if the domain of the quantifier is the set of integers $\mathbb{Z}$, but it is True if the domain is the set of rational numbers $\mathbb{Q}$.

It is often convenient to include the specification of the domain directly with the quantifier, as follows:

$$\begin{aligned} &\exists x \in \mathbb{N}\,(5 < x^2 < 8) \qquad &&\text{(False)},\\ &\exists x \in \mathbb{Q}\,(5 < x^2 < 8) \qquad &&\text{(True)}. \end{aligned}$$

This also applies to $\forall$, and to nested quantifiers:

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{N}\,(y > x)\,. \tag{$*$}$$

(Historical remark: The statement $(*)$ is called the Archimedean property of the real numbers.)

Question: Suppose we know that $P(x) \to Q(x)$ for every x in U. What is the relation between $\{x \in U \mid P(x)\}$ and $\{x \in U \mid Q(x)\}$?

(A) $\{x \in U \mid P(x)\} \subseteq \{x \in U \mid Q(x)\}$

(B) $\{x \in U \mid P(x)\} \supseteq \{x \in U \mid Q(x)\}$

(C) Either one is possible, depending on P and Q

To help us figure this out, let's look at an example. We'll take

$$U = \mathbb{Z}, \qquad P(x) : x > 9, \qquad Q(x) : x > 5.$$

We know $\forall x \in U\,(P(x) \to Q(x))$, and
$\{x \in \mathbb{Z} \mid x > 9\} \subseteq \{x \in \mathbb{Z} \mid x > 5\}$.
So in this case (A) is true. Is (A) always true whenever we know that $P(x) \to Q(x)$ for every x in U? Let's check.

Assume $P(x) \to Q(x)$ for every x in U. Let $y \in \{x \in U \mid P(x)\}$.
Then $P(y)$ is true. (And $y \in U$.) Therefore $Q(y)$ is true.
Therefore $y \in \{x \in U \mid Q(x)\}$.
We conclude that every element of $\{x \in U \mid P(x)\}$ is in $\{x \in U \mid Q(x)\}$.
Therefore $\{x \in U \mid P(x)\} \subseteq \{x \in U \mid Q(x)\}$. This proves that (A) must be true.

Section 2.1.6: Cartesian products

Let A and B be sets. Their Cartesian product, written $A \times B$, is the set of all ordered pairs (c, d) such that $c \in A$ and $d \in B$.

E.g. $\{1, 2, 3\} \times \{2, 3, 4\}$ is the set
$\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$.
Note the contrast between an ordered pair and a set with two elements: for ordered pairs, order matters (e.g., $(2, 3) \neq (3, 2)$), and repetitions are allowed (e.g., $(3, 3)$).

(Warning: You need context to know whether "$(2, 3)$" is meant to be an interval or an ordered pair!)

Similarly, $A \times B \times C$ is the set of all ordered triples (u, v, w) such that $u \in A$, $v \in B$, and $w \in C$;
and, if n is a positive integer,
$A_1 \times A_2 \times \cdots \times A_n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in A_i$ for $i = 1, 2, \ldots, n$.

E.g. $\{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}$
is the set of all 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ where each x_i is 0 or 1.
These can be identified with *bit strings of length 5* (Sec. 1.1.6).

Section 2.1.4: Cardinality

Definition: Let $n \in \mathbb{N}$ and let S be a set. If S has exactly n elements, then we say that n is the *cardinality* of S, and we write $|S| = n$. In this case we also say that S is a *finite set*.

Examples: (a) $\{1, 3, 5, 7, 9\}$ has cardinality 5. It is a finite set.

(b) Is the empty set finite? Yes. It has cardinality 0, and $0 \in \mathbb{N}$.

(c) If A and B are finite sets, then $A \times B$ is finite, and its cardinality is $|A|\,|B|$.

(d) If B is a finite set, and $A \subseteq B$, then A is finite and $|B - A| = |B| - |A|$. (This equation is false if A is not a subset of B.)

An *infinite set* is a set that is not finite. (More about this in Section 2.5.)
Some examples of infinite sets are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

2.2.1: Basic Operations on Sets

The set of elements that are in C and are in D is called the intersection of C and D, and is written $C \cap D$.

Example: Recall our set $A = \{1,3,5,7,9\}$. Let *Odd* be the set of all the nonnegative odd integers. We have

$$\begin{aligned}
\{1,3,5,7,9\} &= \{0,1,2,3,4,5,6,7,8,9,10\} \cap Odd \\
&= \{x \in \mathbb{N} \,|\, x < 11\} \cap \{x \in \mathbb{N} \,|\, x \text{ is odd}\} \\
&= \{x \in \mathbb{N} \,|\, (x < 11) \wedge (x \text{ is odd})\}
\end{aligned}$$

In general, for two predicates G and H,

$$\{x \in U \,|\, G(x)\} \cap \{x \in U \,|\, H(x)\} = \{x \in U \,|\, G(x) \wedge H(x)\}.$$

Similarly, the set of elements that are in C or are in D is called the union of C and D, and is written $C \cup D$.

$$\{x \in U \,|\, G(x)\} \cup \{x \in U \,|\, H(x)\} = \{x \in U \,|\, G(x) \vee H(x)\}.$$

(Notice the visual similarity between symbols $\cup$ and $\vee$, as well as between $\cap$ and $\wedge$.)

Next, consider negation. Suppose $J = \{x \in U \mid P(x)\}$.

How can we describe $\{x \in U \mid \neg P(x)\}$? This is the set of all elements of U that are not in J. We call this the complement of J in U, and write it $\overline{J}$.

Example: What is the complement of $\{1, 2, 3, 4\}$? This question is too vague. To determine the complement, we must specify what the universal set is.

The complement of $\{1, 2, 3, 4\}$ in $\mathbb{N}$ is $\{0, 5, 6, 7, 8, \ldots\}$.
The complement of $\{1, 2, 3, 4\}$ in $\mathbb{Z}$ is
$\{\ldots, -3, -2, -1, 0, 5, 6, 7, 8, \ldots\}$.
The complement of $\{1, 2, 3, 4\}$ in $\mathbb{R}$ is

$$\{x \in \mathbb{R} \mid x < 1 \text{ or } x > 4\} \cup (1, 2) \cup (2, 3) \cup (3, 4).$$

Definition: If A and B are sets, then $A - B$ is the set $\{x \in A \mid x \notin B\}$. This is called the difference of A and B, or the complement of B with respect to A. Note that B does not have to be a subset of A in this definition.

E.g. $$\{1, 3, 5, 7, 9\} - \{1, 2, 3, 4\} = \{5, 7, 9\}.$$

Sec. 2.2.2: Set Identities

Example: Prove $A \cup (B - A) = A \cup B$ (for all sets A and B).

First proof:

$$
\begin{aligned}
A \cup (B - A) &= \{x \in U \mid (x \in A) \vee [(x \in B) \wedge \neg(x \in A)]\} \\
&= \{x \in U \mid [(x \in A) \vee (x \in B)] \wedge \\
&\qquad\qquad [(x \in A) \vee \neg(x \in A)]\} \\
&\qquad \text{(using } p \vee [q \wedge r] \equiv [p \vee q] \wedge [p \vee r]\text{)} \\
&= \{x \in U \mid [(x \in A) \vee (x \in B)] \wedge T\} \\
&\qquad \text{(since } [p \vee \neg p] \equiv T\text{)} \\
&= \{x \in U \mid (x \in A) \vee (x \in B)\} \\
&\qquad \text{(since } [p \wedge T] \equiv p\text{)} \\
&= \{x \in U \mid x \in (A \cup B)\} \\
&= A \cup B.
\end{aligned}
$$

Second proof that $A \cup (B - A) = A \cup B$:

We'll prove this by showing

$$(i)\ A \cup (B - A) \subseteq A \cup B \quad \text{and} \quad (ii)\ A \cup B \subseteq A \cup (B - A)\,.$$

To prove (i): Let $y \in A \cup (B - A)$.
Then $y \in A$ or $y \in (B - A)$. These are two cases to check.
First case: If $y \in A$, then obviously $y \in A \cup B$.
Second case: If $y \in B - A$, then $y \in B$ and $y \notin A$. In particular, $y \in B$. Hence $y \in A \cup B$. This proves (i).

To prove (ii): Let $w \in A \cup B$.
First case: $w \in A$. Then $w \in A \cup (B - A)$ (in fact, $w \in A \cup$ [anything]).
Second case: $w \notin A$. Since $w \in A \cup B$, we must have $w \in B$.

Therefore $w \in \{x \in U \mid x \in B, x \notin A\} = B - A$.

Hence $w \in A \cup (B - A)$. This proves (ii).

This completes the proof that $A \cup (B - A) = A \cup B$. Q.E.D.

Third proof that $A \cup (B - A) = A \cup B$:

We can use a membership table, analogous to a truth table.
It lets us examine all possible cases of whether an element is or is not in A, and again for B.

We use a "1" to indicate that an element is in the set, and a "0" to indicate that it is not.

A	B	$A \cup B$	$B - A$	$A \cup (B - A)$
1	1	1	0	1
1	0	1	0	1
0	1	1	1	1
0	0	0	0	0

Since columns 3 and 5 agree in each row, it follows that
$A \cup (B - A) = A \cup B$.

(Similar idea to a Venn diagram)

Section 2.2.4 describes a way to represent (sub)sets as bit strings.

Example: Among the set of all provinces, let *Salt* be the subset of those that touch salt water.

We can take the universal set to be the set of all ten provinces. We need to specify an order for the members of the universal set. Alphabetical order is a natural choice:

AB, BC, MB, NB, NL, NS, ON, PE, QC, SK.

Then *Salt* contains all the provinces except AB and SK. Thus we can represent *Salt* by the bit string

$$Salt \ : \ 0111111110.$$

The set of all provinces that share a land border with the USA is

$$Land \ : \ 1110001011\,.$$

We find the intersection $Salt \cap Land$ by taking the "bit AND" (Sec. 1.1.6):

$$0111111110 \ \wedge \ 1110001011 \ = \ 0110001010\,.$$

We can take unions of more than two sets at a time.
(Similarly for intersections.)

Example: Let $J_1 = \{1, 2, 3\}$, $J_2 = \{2, 3, 4\}$, $J_3 = \{3, 4, 5\}$, etc.
I.e., for each positive integer n, let $J_n = \{n, n+1, n+2\}$.

Poll: Then $J_3 \cup J_4 \cup J_5 \cup J_6 \; =$

(A) $\{1, 2, 3, 4, 5, 6\}$

(B) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(C) $\{3, 4, 5, 6\}$

(D) $\{3, 4, 5, 6, 7, 8\}$

Poll: How many elements are in $J_3 \cap J_4 \cap J_5 \cap J_6$?

(A) 4

(B) 3

(C) 2

(D) 1

(E) 0

Notation:

$$J_3 \cap J_4 \cap J_5 \cap J_6 = \bigcap_{n=3}^{6} J_n \qquad J_3 \cup J_4 \cup J_5 \cup J_6 = \bigcup_{n=3}^{6} J_n$$

We can also take unions of infinitely many sets. For example, using $J_n = \{n, n+1, n+2\}$ again, we have

$$\begin{aligned} J_1 \cup J_2 \cup J_3 \cup \cdots &= \{1,2,3\} \cup \{2,3,4\} \cup \{3,4,5\} \cup \cdots \\ &= \bigcup_{n=1}^{\infty} J_n \\ &= \{1,2,3,4,5,\ldots\} \qquad = \mathbb{Z}^+. \end{aligned}$$

And

$$\begin{aligned} J_1 \cap J_2 \cap J_3 \cap \cdots &= \{1,2,3\} \cap \{2,3,4\} \cap \{3,4,5\} \cap \cdots \\ &= \bigcap_{n=1}^{\infty} J_n \\ &= \emptyset. \end{aligned}$$

Next class: Read Sections 2.3 and 2.4.

There are two homework items currently posted:
• Homework assignment 2: Questions in Connect, as usual, due Sunday January 29; and
• Problem Set A: Two proofs to write up and submit (by Crowdmark; details to be announced), due Thursday February 2.

The first Midterm Test will be Tuesday February 14.