EECS/MATH 1019
Section 1.8: Introduction to Proofs (continued)

January 24, 2023

Last week, we saw a few different approaches to proving statements.

Direct proofs and indirect proofs
Proof by contraposition (also called contrapositive)
Proof by contradiction
Proof by cases

Today we shall see examples of other kinds of proofs, including existence proofs and uniqueness proofs.

Existence proofs (Section 1.8.3):

Example 1.8.C: Jerry left Ottawa at 1:00 pm and drove to Toronto, 450 km away. He arrived in Toronto at 5:00 pm. Prove that there was a one-hour period during which he drove more than 110 km.

Notice that the conclusion does not say in which hour Jerry drove more than 110 km. We can prove that there must be some such hour.

Example 1.8.C: Jerry left Ottawa at 1:00 pm and drove to Toronto, 450 km away. He arrived in Toronto at 5:00 pm. Prove that there was a one-hour period during which he drove more than 110 km.

The premises are "Jerry left Ottawa at 1:00 pm and drove to Toronto, 450 km away" and "He arrived in Toronto at 5:00 pm." That is, we assume that they are true.

We shall prove the conclusion by contradiction.
Assume that the conclusion is False, i.e. that Jerry drove less than 110 km during each one-hour period.
For $n = 1, 2, 3, 4$, let $D(n)$ be the distance in km that Jerry drove between n:00 pm and $(n + 1)$:00 pm. The above assumption implies that $D(n) \leq 110$ for $n = 1, 2, 3, 4$
Therefore $D(1) + D(2) + D(3) + D(4) \leq 440$. $(*)$
But $D(1) + D(2) + D(3) + D(4)$ is the total distance that Jerry drove, which is 450 (or more, if he took a detour).
This is a contradiction to $(*)$.
Since the contradiction arose from assuming that the conclusion is False, we deduce that the conclusion must be True.

Other examples of existence theorems arise in calculus.
For example, if x is a real variable and $f(x)$ is a polynomial function such that $f(1) = 6$ and $f(2) = 10$, then there must exist a number w somewhere between 1 and 2 such that $f(w) = 7$.

This is a consequence of the Intermediate Value Theorem, which holds for functions that are *continuous*.

(It may seem obvious, but the proof is not simple. See MATH 2001.)

Example 1.8.D: (a) Show that there exists a way to split the numbers $1, 2, 3, \ldots, 12$ into six groups so that the sum of each group is the same. (Each number must be in one and only one of the six groups.)
(b) Is your solution unique?

First, consider part (a). Take a few minutes to think about it. When you think you have solved (a), respond (A) on the **Poll**.

If you solve (a), then you can start thinking about (b).

Example 1.8.D: (a) Show that there exists a way to split the numbers $1, 2, 3, \ldots, 12$ into six groups so that the sum of each group is the same. (Each number must be in one and only one of the six groups.)
(b) Is your solution unique?

Here is a "constructive" existence proof for (a):

$\{1, 12\}$, $\{2, 11\}$, $\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$.

The sum of each group is 13.

What about (b)? Is this the ONLY solution?
If no other solutions come to mind, then this question requires some thought.

Poll:

(A) Yes, I am sure that the solution is unique.
(B) Yes, I think that the solution is unique, but I am not sure.
(C) No, I think the solution is not unique, but I am not sure.
(D) No, I am sure that the solution is not unique.

How could we prove that

$\{1, 12\}$, $\{2, 11\}$, $\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$

is the only way to split up $1, 2, 3 \ldots, 12$ into six groups so that the sum of each group is the same?

One way to prove uniqueness: Program a computer to list all possible ways to split up these numbers into six groups, and verify that in no other way do the all groups have the same sum.
This would be an "exhaustive" proof.
(There are about 10^7 ways. Feasible, but maybe not so smart?)

As a start towards finding a proof, we can think about how to reduce the number of possibilities that such a program would need to check.

How could we prove that

$\{1, 12\}$, $\{2, 11\}$, $\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$

is the only way to split up $1, 2, 3 \ldots, 12$ into six groups so that the sum of each group is the same?

In the above solution, each group has sum 13.
Could another solution have a different sum?

Suppose each of the 6 groups adds up to the same number S.
Then the sum of all the numbers in all the groups is $6S$, which must also equal
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 78$.
Therefore $6S$ must equal 78. That is, S must equal $78/6 = 13$.

This is progress! We have demonstrated that each group must have sum 13.

How could we prove that

$\{1, 12\}$, $\{2, 11\}$, $\{3, 10\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$

is the only way to split up $1, 2, 3 \ldots, 12$ into six groups so that the sum of each group is the same?

We know that any solution must have 6 groups that each add up to 13. How do we complete the proof that the solution is unique? Here are two approaches (argued informally).

First approach: No group can have just one number, since the sum must be 13 and all the numbers are less than 13. Therefore each group has at least 2 numbers. Since $6 \times 2 = 12$, this uses up all the numbers, so no group can have more than 2 numbers.
So each group has exactly 2 numbers. So each group contains 2 numbers that add up to 13. All such groups are on the above list.

How could we prove that

$$\{1,12\},\ \{2,11\},\ \{3,10\},\ \{4,9\},\ \{5,8\},\ \{6,7\}$$

is the only way to split up $1, 2, 3 \ldots, 12$ into six groups so that the sum of each group is the same?

Second approach:

Consider the group that contains the number 12.
Since the group adds up to 13, the group must be $\{12, 1\}$.

This leaves the numbers from 2 to 11 for the remaining 5 groups.

Consider the group that contains the number 11.
Since the group adds up to 13, the group must be $\{11, 2\}$.

This leaves the numbers 3 to 10 for the remaining 4 groups.

Consider the group that contains 10.
(And so on...)

Either of these approaches can be written up to a proper proof.

Example: Prove that $\sqrt{2}+\sqrt{7}$ is irrational. (We know that $\sqrt{2}$ is irrational, but we're not sure whether $\sqrt{7}$ is rational or irrational.)

We shall prove this by contradiction. Let $r=\sqrt{2}+\sqrt{7}$ and assume that r is rational. Since $r>0$, we can write $r=m/n$ for some positive integers m and n.
(Idea: Square both sides of some equation.)

$$\begin{array}{lrcl} \text{We know} & r-\sqrt{2} & = & \sqrt{7} \\ \therefore & r^2-2r\sqrt{2}+(\sqrt{2})^2 & = & (\sqrt{7})^2 \\ \therefore & r^2-2r\sqrt{2}+2 & = & 7. \\ \therefore & r^2+2-7 & = & 2r\sqrt{2} \\ \therefore & \dfrac{r^2-5}{2r} & = & \sqrt{2}\,. \end{array}$$

Plugging in $r=m/n$, we obtain

$$\sqrt{2}=\frac{r^2-5}{2r}=\frac{\frac{m^2}{n^2}-5}{2\,\frac{m}{n}}=\frac{m^2-5n^2}{n^2}\,\frac{n}{2m}\,.$$

But the rightmost expression is rational, and $\sqrt{2}$ is irrational. This gives us a contradiction. We conclude that r is not rational. $\square$

There are many more examples of proofs in the text. Some examples are longer. Try to read them!

In Chapter 5 we will learn about *mathematical induction*, a different method of proof that is well suited to recursive definitions.

We will continue to prove things at various points in the course. With a proof, you can understand why a statement is true; then the statement is more just something that you believe because you were told that it is true.

Next class: Read Sections 2.1 and Section 2.2.

Homework 2 is posted in Connect, due Sunday Jan 29 at 11:59 pm.

In addition, Problem Set A is posted in eClass. It asks you to write up two proofs and submit them by February 2. Submission will be through Crowdmark; details to be announced.