EECS/MATH 1019
Section 1.6: Rules of inference; logical arguments
Section 1.7: Introduction to Proofs

January 17, 2023

1.6: Rules of inference, and logical arguments

This section formalizes the steps in the construction of a logical argument. These ideas will be developed further in EECS/MATH 1090, Introduction to Logic for Computer Science.

We'll look at some examples of logical arguments.

1. Suppose we know the following:

   If the car moves, then it has fuel.

   The car has no fuel.

What can we conclude?

Here is one way to look at the question.

The first statement is (car moves)$\rightarrow$(car has fuel), which can be expressed as [($\neg$ car moves) or (car has fuel)].

The second statement is $\neg$(car has fuel)

So we conclude ($\neg$ car moves), i.e. that the car does not move.

1. Suppose we know the following:

If the car moves, then it has fuel.
The car has no fuel.

Then we can conclude that *the car does not move*.

Let P and Q denote the statements

P : The car moves $\qquad$ Q : The car has fuel.

Then the above argument can be written

$$\begin{array}{rl} & P \to Q \\ & \neg Q \\ \hline \therefore & \neg P \end{array}$$

"modus tollens" rule
This corresponds to the tautology
$[(P \to Q) \wedge (\neg Q)] \to \neg P$.

P	Q	P→Q	$(P \to Q) \wedge \neg Q$	$[(P \to Q) \wedge (\neg Q)] \to \neg P$
T	T	T	F	T
F	F	T	T	T
F	T	T	F	T
T	F	F	F	T

2. Suppose we know the following:

Joan will not go to work if she has high fever.
If Joan has pneumonia, then Joan has high fever.
Joan is at work.

Can we conclude that Joan does not have pneumonia?

Statements:	W:	Joan is at work
	H:	Joan has high fever
	P:	Joan has pneumonia

Steps of the logical argument:

(1)	$H \to \neg W$	Premise
(2)	W	Premise
(3)	$\neg(\neg W)$	Double negation law (Section 1.3)
(4)	$\neg H$	Modus tollens, using (1) and (3)
(5)	$P \to H$	Premise
(6)	$\neg P$	Modus tollens, using (4) and (5).

This completes the proof that Joan does not have pneumonia.

Another rule of inference is

$$\begin{array}{rl} & P \to Q \\ & P \\ \hline \therefore & Q \end{array}$$

The "modus ponens" rule.
This corresponds to the tautology
$[(P \to Q) \wedge P] \to Q$.

This is so obvious that it is easy to forget that it is a rule of inference:

$$\begin{array}{rl} & \text{If Joan has pneumonia, then she has high fever.} \\ & \text{Joan has pneumonia.} \\ \hline \therefore & \text{Joan has high fever.} \end{array}$$

In contrast, the argument

$$\begin{array}{rl} & P \to Q \\ & Q \\ \hline \therefore & P \end{array}$$

is NOT valid.
It is a "fallacy" (Sec. 1.6.6),
but, sadly, we often see it invoked.

E.g. *If COVID is a hoax perpetrated by a secret global conspiracy, then all mainstream media would say that COVID is a real danger. All mainstream media say that COVID is a real danger.*
Therefore *COVID is a hoax perpetrated by a global conspiracy.*
INVALID ARGUMENT

Most of the rules of inference in Table 1 of Section 1.6 are pretty simple. You should know them, but you do not need to memorize their names. For example,

$$\begin{array}{rl} & P \wedge Q \\ \hline \therefore & P \end{array} \qquad \begin{array}{rl} & P \vee Q \\ & \neg P \\ \hline \therefore & Q \end{array} \qquad \begin{array}{rl} & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

Examples from Exercise 4, Section 1.6: What rules are being used?

It is hot today or the pollution is dangerous. It is not hot today. Therefore, the pollution is dangerous.

Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

Steve will work for IBM next summer. Therefore, next summer Steve will work for IBM or he will spend all his time at the beach.

The last one is the rule $\begin{array}{rl} & P \\ \hline \therefore & P \vee Q \end{array}$

There are also rules for quantified statements in Sec. 1.6.7–1.6.8.

Exercise 13(b). Which rules are being used?
Somebody in this class enjoys whale watching.
Everyone who enjoys whale watching cares about ocean pollution.
Therefore, somebody in this class cares about ocean pollution.

Let the domain be the set of people in this class.
Let $W(x)$ be the sentence "x enjoys whale watching."
Let $P(x)$ be the sentence "x cares about ocean pollution."

Premises: (A) $\exists x\, W(x)$, and (B) $\forall x\, (W(x) \to P(x))$.
From (A), $W(y)$ is true for some person y in the class.
(Existential instantiation)
From (B), $W(y) \to P(y)$ for this person y.
(Universal instantiation)
Since $W(y)$ and $W(y) \to P(y)$, we obtain $P(y)$. (Modus ponens)
Since $P(y)$ is true, $\exists x\, P(x)$. (Existential generalization)

This completes the formal proof.

Remarks:

1. The proof on the preceding slide is different (shorter) from the one in the back of the textbook. Upon reflection, there is a minor flaw in the proof we just did. We were given the premise "Everyone who enjoys whale watching cares about ocean pollution", not "Everyone in this class who enjoys whale watching. . . ." It is easy to go from the given to what we need, but it does require one more step.

2. This level of detail may seem excessive, but if you want to teach a computer to reason, then this is the level of detail that you need.

Section 1.7: Introduction to Proofs

A proof is a logical argument that is written to convince the reader that a particular proposition is true.

Each step in a proof should have a clear reason for why it is true.

In this section, and in most of the rest of this course, we shall work with "informal" proofs that are somewhat less constrained than the "formal" logical arguments of Sections 1.1–1.6, where every single step needed justification by an explicit rule or formula chosen from a small list.
We can think of a formal logical proof as something that a machine could read and verify, while an informal proof is something for a human being to read and verify.

An informal proof should be logically complete and correct, and should be written so that the reader can comprehend the overall structure and the details of the logical argument unambiguously. Precision and clarity are essential.

To illustrate, we shall prove the following.

Proposition 1.A. *Let n be an integer. Then n^2 is even if and only if n is even.*

4 is the square of 2. Both are even. (True for $n = 2$.)

9 is the square of 3. Neither is even. (True for $n = 3$.)

16 is the square of 4. Both are even. (True for $n = 4$.)

25 is the square of 5. Neither is even. (True for $n = 5$.)

Is this a proof? Can we use this reasoning to obtain a proof?

It is important to understand the meaning of the statement of the proposition. It can be restated as follows:

Proposition 1.A. *For every integer n, (n^2 is even) $\leftrightarrow$ (n is even).*

Thus, however many squares we list, it will not be enough to prove this for EVERY integer n.

(Besides, to determine whether $(621270328984)^2$ is even or odd, do we really want to have to calculate the value of this square?)

Proposition 1.A. *Let n be an integer. Then n^2 is even if and only if n is even.*

Before we try to find a proof, we need to be clear what we mean by "even."

Definition: An integer m is *even* if there exists an integer k such that $m = 2k$. An integer m is *odd* if there exists an integer k such that $m = 2k + 1$.

Every integer is either even or odd. (Something to think about: Is this obvious?)
E.g. 86 is even because $86 = 2 \times 43$.
-86 is even because $-86 = 2 \times (-43)$.
0 is even because $0 = 2 \times 0$.
83 is odd because $83 = 2 \times 41 + 1$.
-83 is odd because $-83 = 2 \times (-42) + 1$.

Proposition 1.A. *Let n be an integer. Then n^2 is even if and only if n is even.*

Equivalently:

Proposition 1.A. *For every integer n, (n^2 is even) $\leftrightarrow$ (n is even).*

First convince yourself why this is true (or why it is not true.).

Well, if $n = 2k$, then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
And $(2k^2)$ is an integer, so this shows that n^2 is even.

We still need to write this more clearly, but is this reasoning adequate?

Poll:

(A) This reasoning can be made into a full proof.
(B) This reasoning only proves "If n^2 is even, then n is even."
(C) This reasoning only proves "If n is even, then n^2 is even."
(D) This reasoning is not useful for proving the proposition.

Answer: (C).

Proposition 1.A. *Let n be an integer. Then n^2 is even if and only if n is even.*

Recall that $p \leftrightarrow q$ is equivalent to $(p \to q) \land (q \to p)$.
So Proposition 1.A is equivalent to the following:

Proposition 1.A. *Let n be an integer.*
(a) If n is even, then n^2 is even.
(b) If n^2 is even, then n is even.

To prove Proposition, we can prove parts (a) and (b) separately.
We already know the idea behind part (a):

Proof of (a): Assume n is even. Then $n = 2k$ for some integer k. Then

$$n^2 = (2k)^2 = 2(2k^2)\,.$$

Since $n^2 = 2(2k^2)$ and $2k^2$ is an integer, it follows that n^2 is even. This proves part (a).

Now we need to prove part (b).

Proposition 1.A. *Let n be an integer.*
(b) If n^2 is even, then n is even.

First, what is the idea behind (b)? Suppose n^2 is even. Then $n^2 = 2k$ for some integer k. Now what??

Let's try something different. Consider the contrapositive of (b).

Recall $(p \to q) \equiv (\neg q \to \neg p)$.

The contrapositive of (b) is

(b') If n is not even, then n^2 is not even.

Proof of (b'): Assume n is not even. Then n is odd. Therefore $n = 2k + 1$ for some integer k. Then

$$n^2 \;=\; (2k+1)^2 \;=\; 4k^2 + 4k + 1 \;=\; 2(2k^2 + 2k) + 1.$$

Since $n^2 = 2(2k^2 + 2k) + 1$ and $(2k^2 + 2k)$ is an integer, it follows that n^2 is odd. Hence n^2 is not even. This proves (b').

Since (b') is logically equivalent to (b), we have proved (b).

And this completes the proof of Proposition 1.A. Q.E.D.

Next class: Read Section 1.7 and 1.8.1–1.8.2.

Homework 1 is posted in Connect, due Sunday Jan 22 at 11:59 pm.