EECS/MATH 1019
Predicates and Quantifiers (Sections 1.4–1.5)

January 12, 2023

More about Implication

Medical knowledge tells us that any one of the following can occur:

Joan has pneumonia and has a fever,
Joan does not have pneumonia and does not have a fever, or
Joan does not have pneumonia but does have a fever,

and that the following cannot occur:

Joan has pneumonia but does not have a fever.

We can summarize the above knowledge by the sentence

If Joan has pneumonia, then Joan has a fever.

The corresponding truth table is

P: Joan has pneumonia	Q: Joan has a fever	If P then Q
T	T	T
F	F	T
F	T	T
T	F	F

Comment: In its logical usage, the statement "If P then Q" does not include any assumption or suggestion that P causes Q, or that some natural law states that Q is consequence of P, or anything like that. The logical proposition "If P then Q" only tells us that "It not possible that P is true and Q is false."

In other words, the statement "If P then Q" is no more than the contents of the truth table

P	Q	If P then Q
T	T	T
F	F	T
F	T	T
T	F	F

So the following propositions are all TRUE:

(a) If $27 + 38 = 65$, then $27 + 38 + 1 = 66$.
(b) If $27 + 38 = 65$, then $2 + 2 = 4$.
(c) If $27 + 38 = 100$, then $27 + 38 + 1 = 101$.
(d) If $27 + 38 = 100$, then $10000 = 0$.
(e) If $27 + 38 = 100$, then $2 + 2 = 4$.

Here are two more implication statements to consider:

G : If $x > 9$, then $x > 5$. H : If $x > 5$, then $x > 9$.

Comments? As written, these are not propositions, because the truth value depends on x. In spite of this:

Poll: Informally, which of these looks as if it should be valid? Vote (A) for G, or (B) for H.

Write $G(x)$ for the *propositional function* "If $x > 9$, then $x > 5$."
Write $H(x)$ for the *propositional function* "If $x > 5$, then $x > 9$."

Using our definition of truth value for if-then statements, we see for example that
$G(3)$ is True, $G(6)$ is True, $G(10)$ is True,
$H(3)$ is True, and $H(6)$ is False.

In fact, $G(x)$ is true for every real number x, but $H(x)$ is false for some real x.
Thus, the idea behind sentence G above is the following true proposition: "For every real number x, (if $x > 9$ then $x > 5$)."

(This may help clarify why the $p \rightarrow q$ truth table makes sense.)

For each real number x, we defined the sentences

$G(x)$: "If $x > 9$, then $x > 5$", and
$H(x)$: "If $x > 5$, then $x > 9$."

We observed that the following is true:

GG: "For all real numbers x, $G(x)$ is true."

Using the notation "$\forall$", which means "for all" (or "for every"), we can express GG as

GG: $\forall x\, G(x)$.

Similarly, define the statement HH by

HH: $\forall x\, H(x)$. That is, "For all real x, if $x > 5$, then $x > 9$."

We have defined

$G(x)$: "If $x > 9$, then $x > 5$", and
$H(x)$: "If $x > 5$, then $x > 9$."

HH: "$\forall x\, H(x)$." That is, "For all real x, if $x > 5$, then $x > 9$."

But HH is False, because $H(6)$ is false (as are $H(7)$, $H(5.1)$, etc.)
That is, $\neg HH$ is True.
And we know that HH is false because there is at least one x such that $H(x)$ is False. That is,

There exists an x such that $\neg H(x)$ is True. (1)

Using the notation $\exists$, which means "there exists" (or "there is at lest one"), we can write the above sentence (1) as

$\exists x\, (\neg H(x))$.

This statement is in fact logically equivalent to $\neg$HH:

$$(*) \qquad \neg(\forall x\, H(x)) \;\equiv\; \exists x\, (\neg H(x)).$$

The relation $(*)$ is always correct, whatever $H(x)$ is.

We have seen that for *any* propositional function $H(x)$, the equivalence

$$(*) \qquad \neg(\forall x\, H(x)) \;\equiv\; \exists x\,(\neg H(x)).$$

holds. (See subsection 1.4.9.) Similarly, we always have

$$(**) \qquad \neg(\exists x\, H(x)) \;\equiv\; \forall x\,(\neg H(x)).$$

For example, here is how one could find equivalent forms of the (true) statement "There is no number that is less than 2 and greater than 8":

$$\begin{aligned}
\neg[\exists x\,(x < 2 \text{ and } x > 8)] &\equiv \forall x\,[\neg(x < 2 \text{ and } x > 8)] \qquad (\text{by } (**)) \\
&\equiv \forall x\,[\neg(x < 2) \text{ or } \neg(x > 8)] \\
&\qquad\qquad (\text{since } \neg(p \wedge q) \equiv (\neg p) \vee (\neg q)) \\
&\equiv \forall x\,[x \geq 2 \text{ or } x \leq 8]\,.
\end{aligned}$$

An important note on domains

When we write $\forall x$ or $\exists x$, we need to be clear about what possible x's are permissible; that is, what is the *domain* of the quantifier $\forall$ or $\exists$. (See Sections 1.4.4–1.4.5.)

For example, the statement

$$\exists x\,(5 < x^2 < 8)$$

is False if we only allow x to be a natural number or an integer, but it is True if we allow x to be a real number or a rational number (i.e. a fraction). If the domain is not clear, then we could write something like

$$\exists x\,(x \text{ is an integer and } 5 < x^2 < 8)$$

or we could write something like "Let our domain be the set of integers."

(Or, as we shall see in Sec. 2.1.7, writing $\mathbb{Z}$ for the set of integers, "$\exists x \in \mathbb{Z}\,(5 < x^2 < 8)$.")

Another example with predicates and quantifiers

Consider two propositional functions $P(x)$ and $Q(x)$, and the two propositions

$$R: \quad \forall x\, [P(x) \leftrightarrow Q(x)]$$
$$S: \quad [\forall x\, P(x)] \leftrightarrow [\forall x\, Q(x)]$$

Poll: Which of the following is true (for every $P(x)$ and $Q(x)$)?

(A) R is true if and only if S is true.
(B) If R is true, then S is true (but not the converse)
(C) If S is true, then R is true (but not the converse)
(D) None of the above.

It is not hard to find an example that shows that (A) is false. E.g. For the domain of integers: $P(x) : x < 6$ and $Q(x) : x > 2$; or for the domain of Canadian voters, $P(x)$ is "x thinks Justin Trudeau is smart" and $Q(x)$ is "x would vote for Trudeau".
For both of these examples, R is False and S is True (since both sides of S are False).
This also shows that (C) is False. What about (B)?

$$R: \quad \forall x\,[P(x) \leftrightarrow Q(x)]$$
$$S: \quad [\forall x\, P(x)] \leftrightarrow [\forall x\, Q(x)]$$

Is it true that $R \rightarrow S$?
Assume R is True. Then there are two possibilities:
(a) Either $P(x)$ is true for every x, or
(b) There exists at least one x such that $P(x)$ is False.

In case (a), we see from R that $Q(x)$ is also True for every x. Then $\forall x\, P(x)$ and $\forall x\, Q(x)$ are both True. So S is True in case (a).

In case (b), let x_0 be one of the x's such that $P(x)$ is False.
Then we see from R that $Q(x_0)$ is also False.
Then $\forall x\, P(x)$ and $\forall x\, Q(x)$ are BOTH FALSE.
So S is True in case (b).

Combining (a) and (b), we see that S is True in all cases.
We have proved that if R is true, then S is true.

1.5: Nested quantifiers

Example: Our domain will be the set of integers.

For each x, let $K(x)$ be the sentence

$$K(x) : \exists y \, (x < y) \, .$$

E.g., $K(4)$ is the statement "$\exists y \, (4 < y)$." True.

Poll: Which is correct:
(A) $K(x)$ is true for every x
(B) $K(x)$ is never true
(C) $K(x)$ is true for some x but false for other x
(D) The question does not make logical sense

(A) is correct. (For example, whatever x is, you can take y to be $x + 1$, and then $x < y$.)
This shows that "$\forall x \, K(x)$" is a True proposition. That is,

$$\forall x \, [\exists y \, (x < y)] \, .$$

We can omit the square brackets: $\forall x \, \exists y \, (x < y)$.

We have demonstrated that

$$L: \quad \forall x\, \exists y\, (x < y)$$

is True. Can we say whether

$$M: \quad \exists y\, \forall x\, (x < y)$$

is true? Analyze M as we did L. For each y, let $J(y)$ be the sentence

$$J(y) : \forall x\, (x < y)\,.$$

Then M is the sentence $\exists y\, (J(y))$.

Poll: Which is correct:
(A) $J(y)$ is true for every y
(B) $J(y)$ is never true
(C) $J(y)$ is true for some y but false for other y
(D) The question does not make logical sense

Answer: (B). That is, $\neg\exists y\, (J(y))$. So M is False.

Conclusion: We have shown that

$$\begin{aligned} L:&\quad \forall x\, \exists y\, (x < y) \quad \text{is True, but} \\ M:&\quad \exists y\, \forall x\, (x < y) \quad \text{is False.} \end{aligned}$$

That is, <u>the order of quantifiers is important!</u>

A famous quotation:

> *You can fool all of the people some of the time, and some of the people all of the time, but you cannot fool all of the people all of the time.*
> *– Abraham Lincoln (16th president of the USA)*

Let's try to express this with quantifiers.

Let $L(x, t)$ be the sentence "You can fool x at time t." Here the domain of x is the set of all people, and the domain of t is the set of all times.

You cannot fool all of the people all of the time: $\neg[\forall x\, \forall t\, L(x, t)]$.

You can fool some of the people all of the time:
Poll: Which expression best describes this part?

(A) $\forall x\, \exists t\, L(x, t)$

(B) $\exists x\, \forall t\, L(x, t)$

(C) $\forall t\, \exists x\, L(x, t)$

(D) $\exists t\, \forall x\, L(x, t)$

Answer: (B)

You can fool all of the people some of the time, and some of the people all of the time, but you cannot fool all of the people all of the time.
– Abraham Lincoln

$L(x, t)$ is the sentence "You can fool x at time t."

You can fool all of the people some of the time.

There are two plausible interpretations for this:
(A) $\forall x \, \exists t \, L(x, t)$
(B) $\exists t \, \forall x \, L(x, t)$

Poll: Which seems like the better interpretation?

(A) says that each person has some time when they can be fooled.
(B) says that there is some time when everyone can be fooled.

Assuming that Mr. Lincoln intended (A), then we have:

$$[\forall x \, \exists t \, L(x, t)] \ \wedge \ [\exists x \, \forall t \, L(x, t)] \ \wedge \ \neg[\forall x \, \forall t \, L(x, t)].$$
– Abraham Lincoln

Important: In addition to attending lectures, you are expected to **read the textbook**. It would be a poor use of time for me as instructor to repeat everything that you could be reading in the book. I tend to spend class time on the parts of the text that are most fundamental, and/or more difficult to understand.

Next class: We shall discuss Section 1.6, and may start Section 1.7.

Homework 1 is posted in Connect, due Sunday Jan 22 at 11:59 pm. It covers Sections 1.1–1.6.