# EECS 1019 Fall 2023-24 – Midterm 2

**Instructor: Jeff Edmonds**

Keep your answers short, clear, and readable.
Do not repeat the question.

1. **Simplifying Parity (Purple Table):** Suppose I tell you that $\alpha$ is true. I claim that this means that everywhere $\alpha \oplus \beta$ appears it can be simplified to $\neg\beta$.
   Give a purple table proof of $\alpha \rightarrow [\neg\beta \rightarrow [(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]]$.
   10 marks, 9 sentences

   - Answer:

     1) Deduction Goal: $\alpha \rightarrow [\neg\beta \rightarrow [(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]]$
     2) $\alpha$ — Assumption/Premise
     3) Deduction Goal: $[\neg\beta \rightarrow [(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]]$
     4) $\neg\beta$ — Assumption/Premise
     5) $(\alpha\wedge\neg\beta)$ — Building/Eval And (2 & 4)
     6) $(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]$ — Building/Eval Or (5)
     8) $[\neg\beta \rightarrow [(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]]$ — Conclude deduction.
     9) $\alpha \rightarrow [\neg\beta \rightarrow [(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)]]$ — Conclude deduction.

   - Answer: During the test, I said they could do this instead:

     1) Deduction Goal: $\alpha \rightarrow [[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \rightarrow \neg\beta]$
     2) $\alpha$ — Assumption/Premise
     3) Deduction Goal: $[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \rightarrow \neg\beta$
     4) $(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)$ — Assumption/Premise

        The purple table gives two ways to use an OR: "Selecting Or" and "Cases".
        In "Cases", the first case would be $(\alpha\wedge\neg\beta)$ and the second $(\neg\alpha\wedge\beta)$.
        The problem with doing this is that that the second case leads to a contradiction.
        It is not clear what to do with that.
        In "Selecting Or", we prove that the first is true by proving the second is false. Lets do that.
        We know $\alpha$ is true. Hence, $(\neg\alpha)$ is false and so is $(\neg\alpha\wedge\beta)$.
        "Building/Eval And" lets you do that, i.e. from $\neg\beta$ conclude $(\beta\wedge\gamma)$ is false.
     5) $\neg(\neg\alpha\wedge\beta)$ — Building/Eval And (2)

        Having proved the second case is false, we know that the first is true.
     6) $(\alpha\wedge\neg\beta)$ — Selecting Or (4&5)

        As would have happened in "Cases", the first case gives us what we want.
     7) $\neg\beta$ — Separating And (6)
     8) $[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \rightarrow \neg\beta$ — Conclude deduction.
     9) $\alpha \rightarrow [[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \rightarrow \neg\beta]$ — Conclude deduction.

2. Here is a question the book proved by contradiction. Jeff NOT does like proof by contradiction because it turns things around unnecessarily. Also it is not constructive.

You are now to prove it using the prover adversary game.
Prove that every set $S = \{a_1, a_2, \ldots, a_n\}$ of numbers contains a number that is at least the average of the numbers, namely
$\forall$ *sets* $S$, $\exists x \in S, x \geq a_{avg}$.
Hint: Let $a_{avg} = \frac{1}{n}[a_1+a_2+\ldots+a_n]$ denote the average.
Hint: Let $a_{max}$ denote the maximum in the set, i.e., that which is at least as big as each of the $a_i$.
Hint: Just play the game. There are three clear lines in the proof. What are they?
10 marks, 4 sentences

- Answer:
  1) Let $S$ be an arbitrary set of values.
  2) Let $x$ be $a_{max}$, its maximum value.
  3) $a_{avg} = \frac{1}{m}[a_1+a_2+\ldots+a_n] \leq \frac{1}{n}[a_{max}+a_{max}+\ldots+a_{max}] = \frac{1}{n} \times n \times a_{max} = a_{max}$
  Since the prover wins, it must be true.

(deleted question) Let $S^+ = \{x \in Reals \mid x > 0\}$ denote the set of positive reals. Explain what the following means and give a purple table proof of it: $\forall x \in S^+, \exists y \in S^+, y < x$.
10 marks, 5 sentences

- Answer: Each $x$ fails to be the minimum in $S^+$ because there is a $y$ in $S^+$ that is smaller. Hence, the set $S^+$ has no minimum $x$.
  Proof:
  - Let $x$ be an arbitrary value in $S^+$.
  - Let $y = \frac{1}{2}x$,
  - Note $y \in S^+$ and $y < x$.
  - Hence $x$ (and all values) is not the minimum value in $S^+$.

3. **Growth of Functions and $\Theta$:**

   (a) Put numbers 1,2,3,... in front of the following sorting them with 1 being the smallest and ? being the largest as $n$ grows arbitrarily large?
   Explanation not needed. 10 marks

   - 3 $5 + sin(n^2)$
   - 8 $(1.01)^n$
   - 1 $(0.9)^n$
   - 5 $\log_{10}(10^n)$
   - 6 $n^2 \log^{100} n$
   - 7 $n^{2.01}$
   - 4 $\sqrt{n}$
   - 2 $\frac{1}{n}$

   - 1 $(0.9)^n$: Shrinking exponentially
   - 2 $\frac{1}{n}$: Shrinking polynomially
   - 3 $5 + sin(n^2)$: Considered "constant" because it is between 4 and 6
   - 4 $\sqrt{n}$: $n^{\frac{1}{2}}$.
   - 5 $\log_{10}(10^n)$: $= n$
   - 6 $n^2 \log^{100} n$: Considered "polynomial" because it is between $n^2$ and $n^{2.01}$.
   - 7 $n^{2.01}$: the $n^{.01}$ dominates the $\log^{100} n$ because the *log* grows very slowly.
   - 8 $(1.01)^n$: Grows exponentially like a bank account with 1% interest. In 100 years, your money will double (actually times $e = 2.71$).

   (b) Sorry. My bad. What Practice 2 said was wrong when it said one of the following two statements: "Functions in $\Theta(1)$ might go up and down with $n$ as long as they:
   - (I) don't go negative or above a million."
   - (II) don't go below some constant like 0.01 or above some constant like a million."

   Argue which is the correct statement by considering the function $f(n) = \frac{3}{n}$.
   Argue whether or not $f(n) = \Theta(1)$. Why?

Argue whether (I) or (II) or both or neither is true for $f(n)$. Why?
Argue whether "$f(n) = \Theta(1)$ iff (I)" or "$f(n) = \Theta(1)$ iff (II)" is wrong or (mostly) correct. Why?
10 marks, 5 sentences

- Answer: $f(n) = \frac{3}{n}$ is $\Theta(\frac{1}{n})$ not $\Theta(1)$ because it is shrinking.
  The first statement is true because $f(n)$ does not go negetive, but the second is false becaues it does go below every constant like 0.01.
  The first, which was in practice 2, is wrong besause the $f(n) = \Theta(1)$ is false and (I) is true.
  The second is (mostly) correct. It fails to mention $n_0$. We do need $n \geq 1$.
  The (II) is also better than (I) because it does not fix the upper constant to be a million.

(c) Which functions are considered to be a reasonable running time for an algoirthm. Give both the name of the class of functions and how to write it using at $\Theta$.
5 marks, 1 sentences

- Answer: Functions in $n^{\Theta(1)}$ grow polynomially and are considered a reasonable running time.

(d) Give the simplest $\Theta$, $\mathcal{O}$, $\Omega$ and $?^{\Theta(?)}$ of $f(n) = 5 \cdot n^{5.5} \log^{100}(n) + 1000n^4$.
5 marks, 1 sentences

- Answer: $f(n) = \Theta(n^{5.5} \log^{100}(n))$, $\mathcal{O}(n^{5.6})$, $\Omega(n^{5.5})$, $n^{\Theta(1)}$.

(e) Approximate the sum $\Theta(\sum_{i=1}^{n} f(i)) = \Theta(\sum_{i=1}^{n} i^4)$. How did you get this?
5 marks, 3 sentences

- Answer: The terms grow polynomially, which is "closer" to arithmetic than to geometric. Half the terms are within a constant of the biggest term.
  Hence, $\sum_{i=1}^{n} f(i) = \Theta(\# \text{ of terms} \cdot \text{last term}) = \Theta(n \cdot f(n)) = \Theta(n^5)$.

(f) Recall that $f(n) = \mathcal{O}(g(n))$ is defined as $\exists c, \exists n_0, \forall n \geq n_0, f(n) \leq cg(n)$.
Use this definition to prove $f(n) = 5n^9 - 4n^3 + 7n^3 + 9$ and $g(n) = n^9$.
10 marks, 4 sentences

- Answer:
  i. Let $c = 5 + 7 + 9$ and $n_0 = 1$.
  ii. Let $n \geq 1$ be arbitrary.
  iii. $f(n) = 5n^9 - 4n^3 + 7n^3 + 9 \leq 5n^9 + 7n^9 + 9n^9 \leq cn^9$
  iv. Hence, it is true.

4. **Finite/Counting/Infinite:** You get to heaven. It is beautiful with an infinite number of types of flowers. You over hear God telling the gardener to be sure to water the flower RoseyRoseRose. When you ask, you are assured that the gardener can identify each and every flower in this way. Is the number of flowers Finite/Countable/Infinite? How do you know?
5 marks, 5 sentences

- Answer: Countably Infinite. The implication is that each flower has a finite name that identifies it. Jeff assured us that if every item in a set has a finite identifying name then the number of them is countably infinite. The set of finite names is effectively the same as the set of finite length ASCII strings which effectively can be viewed as the set of integers which by definition is countable. If each of flowers is assigned a unique one of these then there cant be more flowers than integers.

5. **Structural Induction & Linear Recurrences:**

(a) In general, how does strong (structual) induction work?
5 marks, 3 sentences

- Answer: By way of strong (structural) induction, we understand the structure and size of $A_i$ for $i < n$. We learn the structure and size of $A_n$ as follows: Make $A_n$ into smaller $A_i$ which we know about. Use this information about $A_i$ to learn about $A_n$.

(b) Let $A_n = \{s \in \{0,1,2,3,4,5\}^* \mid cost(s) = n\}$ denote the set of strings of 0, 1, 2, 3, 4, and 5 that have cost $n$, where the cost is 1 per character 0, 1, 2, and 3 and is 2 per character 4 and 5. For example, $cost(02514) = 1+1+2+1+2 = 7$.
Give the recurrence relation giving the number of strings in the set $A_n$. No explanation needed.
5 marks, 1 sentences

- Answer: $|A_n| = 4|A_{n-1}| + 2|A_{n-2}|$

6. **Recurrence Relations:**

(a) Suppose we are looking for the family of all closed formulas $a_n = f(n)$ that satisfy the linear recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ when you don't know $a_0$ or $a_1$.
What is your guess of a possible solution $a_n = f(n)$?
From this, how do you get to the quadratic $r^2 - r - 6 = 0$?
10 marks, 4 sentences

- Answer: The guess is $a_n = r^n$ for some constant $r$.
Plugging this in gives $r^n = r^{n-1} + 6r^{n-2}$.
Dividing by $r^{n-2}$ gives $r^2 = r + 6$.
Rearranging gives the quadratic $r^2 - r - 6 = 0$?

(b) Suppose $r^2 - r - 6 = (r-3)(r+2) = 0$ has roots $r_1 = 3$ and $r_2 = -2$.
Give the closed form equation for $a_n = f(n)$ in terms of unkown constants $\alpha_1$ and $\alpha_2$.
No explantion needed.
5 marks, 1 sentences

- Answer: $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot (-2)^n$

(c) What determines the correct values of $\alpha_1$ and $\alpha_2$ for your equation $a_n = f(n)$?
5 marks, 1 sentences

- Answer: Knowing $a_0 = f(0)$ and $a_1 = f(1)$, lets us solve for $\alpha_1$ and $\alpha_2$.