EECS 1019 Fall 2023-24 – Midterm 1

Instructor: Jeff Edmonds

Keep your answers short, clear, and readable.
Do not repeat the question.
T is for *True*, F is for *False*. $\neg$ is *Not*, $\wedge$ is A for *And*, $\vee$ is *OR*, $\rightarrow$ is *Implies*, $\oplus$ is *Parity*,
$\forall$ is A for *forAll*, $\exists$ is E for *Exists*.

Each such rule (in the purple table) has the form: From α & β conclude γ.
- If you already have lines in your proof of the form α & β,
- then you can add the line of the form γ to your proof.

Recall, a proof is a sequence of statements, where each statement is
- either an axiom, i.e., known to be true
- or follows from previous lines using some rule from the purple table (or the book).

Number the lines of your proof 1, 2, 3, ...
For each, give the name of the rule you use to prove that line and the line numbers of any previous lines used.
Do not skip steps. (Except for dropping $\neg\neg$)
Be sure to indent appropriately.
Multiple Choice. Which sentence relates best to the given English?
Answer on the back bubble page.

1. Would that be fries or salad?: a) $p \wedge q$; b) $p \vee q$; c) $p \oplus q$; d) other
   - Answer: c
2. p: a) T/F variable; b) T/F sentence; c) object; d) other
   - Answer: a
3. Converse of $p \rightarrow q$: a) $\neg p \rightarrow \neg q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; d) a & b; e) other
   - Answer: d
4. Image of $p \rightarrow q$: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other
   - Answer: e
5. p whenever q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other
   - Answer: b
6. p follows from q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other
   - Answer: b
7. q is necessary for p: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other
   - Answer: c
8. Can I have anything?: a) $\forall x$ I-can-have(x); b) $\exists x$ I-can-have(x); c) other
   - Answer: a
9. Is there anything I can have?: a) $\forall x$ I-can-have(x); b) $\exists x$ I-can-have(x); c) other
   - Answer: b
10. Everyone loves her: a) $\forall x\ \exists y\ loves(x, y)$; b) $\exists y\ \forall x\ loves(x, y)$; c) other

- Answer: b

11. Every Hindu has God: a) $\forall x\ \exists y$; b) $\exists y\ \forall x$; c) other
    - Answer: a

12. Only Jeff would: a) $would(\text{Jeff}) \wedge \forall x \neq \text{Jeff}\ \neg would(x)$; b) $would(\text{Jeff}) \wedge \exists x \neq \text{Jeff}\ \neg would(x)$; c) $would(\text{Jeff}) \vee \forall x \neq \text{Jeff}\ negwould(x)$; d) other
    - Answer: a

13. Would Jeff or anybody do the dishes: a) $would(\text{Jeff}) \wedge \forall x \neq \text{Jeff}\ \neg would(x)$; b) $would(\text{Jeff}) \vee \forall x \neq \text{Jeff}\ would(x)$; c) $would(\text{Jeff}) \vee \exists x \neq \text{Jeff}\ would(x)$; d) $\exists x\ would(x)$; e) c & d
    - Answer: e

1. $[A \rightarrow B] \rightarrow [B \rightarrow A]$
   (a) (Deleted:) If valid, i.e.. true in every setting, what would it mean?
   - Answer: It says that for every implication, the converse is also true.

   (b) (Deleted:) In class, Jeff demonstrates that this is not always true by giving a counter example involving objects in our daily life. Give that example or, if you can't remember it, make up another.
   - Answer: $[Hound \rightarrow Dog] \rightarrow [Dog \rightarrow Hound]$

   (c) (3 marks) Give an assignment to A and B under which this expression evaluates to false.
   Do this by forming a tree of T/F. Under the sentence below, write T or F under each variable. Below that write T or F for each [...]. Below this write F for the entire expression.
   $[A \rightarrow B] \rightarrow [B \rightarrow A]$
   - Answer: Start by thinking of a setting of X and Y for which $[X \rightarrow Y]$ is false.
     It is easier to do this backwards, i.e., do the last F, then for [], then for A and B.
     $[A \rightarrow B] \rightarrow [B \rightarrow A]$
     $[F \rightarrow T] \rightarrow [T \rightarrow F]$
     $[T] \rightarrow [F]$
     F

   (d) (4 marks) There is a rule that translates $[X \rightarrow Y]$ into the OR/$\vee$ expression $[??\vee??]$.
   Use this rule to translate each of the $\rightarrow$ into Or/$\vee$.
   Then use other rules to put it into the easier of:
   - Conjunctive Normal Form $(A \vee \neg B \vee \neg C) \wedge (\neg D \vee E \vee F)$
   - Disjunctive Normal Form $(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E \wedge F)$.
   You might want to check that under the setting from the previous question, the statement is still false.
   - Answer: $[A \rightarrow B] \rightarrow [B \rightarrow A]$
     $[\neg A \vee B] \rightarrow [\neg B \vee A]$
     $\neg[\neg A \vee B] \vee [\neg B \vee A]$
     $[A \wedge \neg B] \vee \neg B \vee A$
     $[F \wedge \neg T] \vee \neg T \vee F$
     $F \vee F \vee F = F$

2. (20 marks) *Resolution* plays an important role in AI and is used in Prologue.
   It's *Cut Rule* corresponds to the statement: $S \equiv [[(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]]$.
   For each rule listed below, if you can complete the proof of S using the rule, then you **DO NOT** need to do anything. If you can't see how to use the rule in the proof of S, then minimally state what the rule is and if you can state what α, β, and γ would be in the proof of S.
   (a) Deduction (b) Separating And (c) Proof by cases (d) Excluded Middle (e) Selecting Or (f) Building/Eval Or

(a) Deduction

- Answer: Deduction (Only needed if not used in proof):
You say "Goal is to prove $\alpha \rightarrow \beta$ by deduction"
You indent. Within this indenting, you assume the α.
From this your prove the β. Then you stop the indenting and conclude: $\alpha \rightarrow \beta$.
Reason NOT NEEDED: It is true because if α is false, them $\alpha \rightarrow \beta$ is automatically true. Hence, we only need to consider the case in which α is true. In this case, for $\alpha \rightarrow \beta$ to be true, we need β to be true.
For S, $\alpha \equiv [[(\neg p \vee r) \wedge (p \vee q)]$ and $\beta \equiv [r \vee q]]$.

(b) Separating And

- Answer: Separating And (Only needed if not used in proof):
From $\alpha \wedge \beta$, conclude α. (Conclude β too if you want.)
Reason NOT NEEDED: It is true because $\wedge$ means that they are each separately true.
For S, $\alpha \equiv (\neg p \vee r)$ and $\beta \equiv (p \vee q)]$

(c) Proof by cases
Hint: Our cases will be "p is true" and "p is false. What is your γ?
Hint: Be sure that you prove all three steps needed for this rule before concluding.

- Answer: Proof by cases (Only needed if not used in proof):
You must initially prove the following yourself:
- 1) $\alpha \vee \beta$, i.e., at least one of the cases is true.
- 2) $\alpha \rightarrow \gamma$, i.e., If the first case is true, you can prove γ.
- 3) $\beta \rightarrow \gamma$, i.e., If the second case is true, you can prove γ.
- Then you can conclude γ is true.
Reason NOT NEEDED: It is true because, there are only two case and either way γ is true.
For S, $\alpha \equiv p$, $\beta \equiv \neg p$, and $\gamma \equiv [r \vee q]]$.

(d) Excluded Middle

- Answer: Excluded Middle (Only needed if not used in proof):
The purple table has two of these. $\alpha \vee \neg\alpha$ and $\neg(\alpha \wedge \neg\alpha)$.
Reason NOT NEEDED: It is true because at α can't be in middle. It is either true or false, and not both
For S, we need cases $(p \vee \neg p)$.

(e) Selecting Or

- Answer: Selecting Or (Only needed if not used in proof):
From $\alpha \vee \beta$ and $\neg\alpha$, conclude β.
Reason NOT NEEDED: It is true because $\vee$ means that at least one of these is true. If is not α and it must be β.
For S we use it twice:
$\alpha \equiv p$ and $\beta \equiv r$.
$\alpha \equiv \neg p$ and $\beta \equiv q$.

(f) Building/Eval Or

- Answer: Building/Eval Or (Only needed if not used in proof):
From α, conclude $\alpha \vee \gamma$.
Reason NOT NEEDED: It is true because $\vee$ means that at least one of these is true. If we already know that α is true, then we are done.
For S we use it twice:
From r, build $[r \vee q]$ and
From q, build $[r \vee q]$

You will now give the proof of $S = [(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]$.
Hint: My proof uses the above rules in the given order. My proof has 14 lines.
Hint: Our cases will be "p is true" and "p is false. What is your γ?

- Answer:

1)	Deduction Goal: $[(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]$	
2)	$(\neg p \vee r) \wedge (p \vee q)$	Assumption/Premise
3)	$\neg p \vee r$	Separating And
4)	$p \vee q$	
5)	Cases Goal: $r \vee q$. Cases p and $\neg p$.	
6)	$p \vee \neg p$	Excluded Middle
7)	Case p:	Assumed by cases/deduction.
8)	r	Selecting Or from 3 and 7
9)	$r \vee q$	Building/Eval Or from 8
10)	Case $\neg p$:	Assumed by cases/deduction.
11)	q	Selecting Or from 4 and 10
12)	$r \vee q$	Building/Eval Or from 11
13)	$r \vee q$	Conclude cases 6, 9, & 12
14)	$[(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]$	Conclude deduction.

3. Recall that if function g maps $g(0) = 5$, $g(1) = 2$, $g(2) = 4$, $g(3) = 2$, then $g(\{0,1,2,3\}) = \{2,4,5\}$. Consider a function f that maps every positive integer in $\{0,1,2,\ldots\}$ to $\{0,1,2,\ldots\}$. Suppose I told you that $f(\{0,1\}) = \{0\}$, $f(\{0,1,2\}) = \{0,1\}$, $f(\{0,1,2,3\}) = \{0,1,2\}$, and $\forall i \geq 1$, $f(\{0,1,2,3,\ldots,i\}) = \{0,1,2,\ldots,i{-}1\}$.

   (a) (4 marks) Fill in as many values of $f(x)$ that you can.

   (b) (2 marks) What is $f \circ f \circ f(\{0,1,2,3,4,5\})$? No explanation needed.

   - Answer:

x	0	1	2	3	4	5	x
$f(x)$	0	0	1	2	3	4	$x{-}1$

   - Answer: $\{0,1,2\}$

4. Quantifiers over the reals.

   (a) (3 marks) What does the sentence $\forall x\ \exists y\ [x{\times}y{=}1]$ mean?
   Do NOT say "Forall x, there is a y, ... — Zero marks.
   Instead, what property does it attribute to real values x?
   Is the statement true?
   If not what is a counter example?
   Are there more than one counter examples?

   - Answer: It says that every real value x has a multiplicative inverse $\frac{1}{x}$.
     It is not true because it is not true for the single counter example $x = 0$.

   (b) (3 marks) What does the sentence $\exists a\ \forall x\ \exists y\ [[x{=}a] \vee [x{\times}y{=}1]]$ say about the real values a?
   Is it true? If so, what is a?
   Hint: The OR/$\vee$ part may be particularly confusing. Ignore it.
   Guess. What key role do you think a will play in the discussion we are having?

   - Answer: It is true. It says that every real value x has a multiplicative inverse except for the one counter example $a = 0$.

   (c) (10 marks) Prove that the sentence is true under the reals:
   $\exists a\ \forall x\ \exists y\ [[x{=}a] \vee [x{\times}y{=}1]]$
   Don't panic. Just play the game. Who gives which objects and in what order?

Hint: There will be two cases. Say "If case ...., then ...., else, ...".
Hint: Use the notation $c_\forall$ and $c_\exists$.
Hint: If you don't know which object to give, I personally like the object 5.

- Answer: Prove $\exists a\ \forall x\ \exists y\ [[x=a] \vee [x \times y=1]]$
  Let $a_\exists = 0$ (constructed by the prover)
  Let $x_\forall$ be an arbitrary real number (given to us by the adversary)
  If $x_\forall \neq 0$, then let $y_\exists = \frac{1}{x_\forall}$
  Else let $y_\exists = 5$ (a fine number constructed by the prover)
  Now we need to verify $[[x_\forall = a_\exists] \vee [x_\forall \times y_\exists = 1]]$
  If $x_\forall \neq 0$, then $x_\forall \times y_\exists = x_\forall \times \frac{1}{x_\forall} = 1$
  Else $x_\forall = 0 = a_\exists$
  Either way the OR is true.
  Prover can always win. Hence, the statement is true.
  "Every real value x has a multiplicative inverse except for the one counter example $a = 0$."