EECS 1019 – Practice 2

Instructor: Jeff Edmonds

To be (computer) marked as the 10%.

Instruction for not Cheating:
- Truly my goal is to maximize your learning for other tests.
- Read all available material and get your friends, chat-gpt, and Jeff to explain the QUESTIONS to you.
- Try answering them yourself - but get more explanation if you really need.
- Try not to memorize the answers.
- Do something mindless for at least an hour to minimize memorization and then enter the answers by yourself without help or notes.

Simplifying Parity (Purple Table): Suppose I tell you that α is true. I claim that this means that everywhere $\alpha \oplus \beta$ appears it can be simplified to $\neg\beta$. Let's consider whether this is true.

1. What does $\alpha \oplus \beta$ NOT means?

   (a) At least one of these is true.

   (b) Exactly one of these is true.

   (c) We add them together in binary where $1+1=0$.

   (d) $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$.

   (e) All correct meanings.

2. Assuming α, which of the following is NOT true?

   (a) $\alpha \oplus \beta$ is true iff β is false.

   (b) β is true then $\alpha \oplus \beta$ is false.

   (c) β must be true.

   (d) $\alpha \oplus \beta$ iff $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$.

   (e) All true

3. Which of the following is NOT a good and relevant proof?

   (a)

1) Deduction Goal: $\alpha \rightarrow [(\alpha \oplus \beta) \equiv \neg\beta]$	
2) α	Assumption/Premise
3) Proof by Cases of $(\alpha \oplus \beta) \equiv \neg\beta$	
4) Case β is true	
5) $\alpha \oplus \beta = T \oplus T = F$	Definition of $\oplus$
6) Case β is false	
7) $\alpha \oplus \beta = T \oplus F = T$	Definition of $\oplus$
8) $\beta \vee \neg\beta$	Excluded Middle
9) $(\alpha \oplus \beta) \equiv \neg\beta$	Cases 5,7,8
10) $\alpha \rightarrow [(\alpha \oplus \beta) \equiv \neg\beta]$	Conclude deduction.

   (b)

1) Deduction Goal: $\alpha \rightarrow [\neg\beta \rightarrow [(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]]$	
2) α	Assumption/Premise
3) Deduction Goal: $[\neg\beta \rightarrow [(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]]$	
4) $\neg\beta$	Assumption/Premise
5) $(\alpha \wedge \neg\beta)$	Building/Eval And (2 & 4)
6) $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]$	Building/Eval Or (5)
8) $[\neg\beta \rightarrow [(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]]$	Conclude deduction.
9) $\alpha \rightarrow [\neg\beta \rightarrow [(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]]$	Conclude deduction.

(c)

1) Deduction Goal: $\alpha \to [[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \to \neg\beta]$
2) α — Assumption/Premise
3) Deduction Goal: $[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \to \neg\beta$
4) $(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)$ — Assumption/Premise

The purple table gives two ways to use an OR: "Selecting Or" and "Cases".
In "Cases", the first case would be $(\alpha\wedge\neg\beta)$ and the second $(\neg\alpha\wedge\beta)$.
The problem with doing this is that that the second case leads to a contradiction.
It is not clear what to do with that.
In "Selecting Or", we prove that the first is true by proving the second is false. Lets do that.
We know α is true. Hence, $(\neg\alpha)$ is false and so is $(\neg\alpha\wedge\beta)$.
"Building/Eval And" lets you do that, i.e. from $\neg\beta$ conclude $(\beta\wedge\gamma)$ is false.

5) $\neg(\neg\alpha\wedge\beta)$ — Building/Eval And (2)

Having proved the second case is false, we know that the first is true.

6) $(\alpha\wedge\neg\beta)$ — Selecting Or (4&5)

As would have happened in "Cases", the first case gives us what we want.

7) $\neg\beta$ — Separating And (6)
8) $[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \to \neg\beta$ — Conclude deduction.
9) $\alpha \to [[(\alpha\wedge\neg\beta)\vee(\neg\alpha\wedge\beta)] \to \neg\beta]$ — Conclude deduction.

(d) All a, b, and c are great and relevant proofs.

Minimum Value (Forall Exists Game): Let $S^+ = \{x \in Reals \mid x > 0\}$ denote the set of positive reals.

4. Which is true about the minimum value in S^+?
   (a) It is like 0.00001 but has an infinite description.
   (b) It is zero.
   (c) It does not exist.
   (d) It is 1.

5. Which of these say something different?
   (a) $\exists x \in S^+,\ \forall y \in S^+, x \leq y$.
   (b) $\forall x \in S^+, \exists y \in S^+, y < x$.
   (c) x is the minimum in S^+.
   (d) x is in S^+ and other values in S^+ are bigger.
   (e) They all say the same thing.

6. What is the first line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
   (a) Let x be an arbitrary value in S^+.
   (b) Let x be 0.
   (c) Let x be the minimum value in S^+.
   (d) By way of contradiction, assume $x \geq y$.
   (e) Let y be an arbitrary value in S^+.

7. What is the second line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
   (a) Let $y = \frac{1}{2}x$,
   (b) Let y be 0.

(c) Let y be the minimum value in S^+.

(d) Let x be an arbitrary value in S^+.

(e) Let $y = x + 1$.

8. What is the third line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?

   (a) $y \in S^+$ and $y < x$.

   (b) This gives a contradiction.

   (c) y is the minimum value in S^+.

   (d) y is not the minimum value in S^+.

   (e) x is the minimum value in S^+.

9. What is the fourth line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?

   (a) Hence x (and all values) is not the minimum value in S^+.

   (b) Hence x is the minimum value in S^+.

   (c) Hence by way of contradiction, the opposite must be true.

   (d) This has nothing to do with minimums.

Growth of Functions and Θ:

10. Which function is the smallest as n grows arbitrarily large?

    (a) $(0.9)^n$

    (b) $(1.01)^n$

    (c) $\sqrt{n}$

    (d) $\log n$

    (e) $5 + sin(n^2)$

11. Which function is the largest as n grows arbitrarily large?

    (a) $(0.9)^n$

    (b) $(1.01)^n$

    (c) n^{10000}

    (d) $\log(10^n)$

    (e) $5 \cdot n^{5.5} \log^{100}(n)$

12. Which is NOT true?

    (a) Functions in $\Theta(1)$ might go up and down with n as long as they don't go below 0.01 or above a million.

    (b) Functions in $2^{\Theta(n)}$ grow exponentially and are considered an unreasonable running time.

    (c) Functions in $\log^{\Theta(1)}(n)$ grow exponentially.

    (d) Functions in $n^{\Theta(1)}$ grow polynomially and are considered a reasonable running time.

    (e) All of the above

13. $f(n) = 5 \cdot n^{5.5} \log^{100}(n) + 1000n^4$. Which is NOT true?

    (a) $f(n) = \Theta(n^{5.5})$

    (b) $f(n) = \Theta(n^{5.5} \log^{100}(n))$

    (c) $f(n) = \mathcal{O}(n^{5.6})$

(d) $f(n) = \Omega(n^{5.5})$

(e) $f(n) = n^{\Theta(1)}$.

14. Approximate the sum $\Theta(\sum_{i=1}^{n} f(i)) = \Theta(\sum_{i=1}^{n} i^3)$. Which IS true?

    (a) The terms grow polynomially, which is "closer" to arithmetic than to geometric. Half the terms are approximately within a constant of the biggest term.
    Hence, $\sum_{i=1}^{n} f(i) = \Theta(\#\text{ of terms} \cdot \text{last term}) = \Theta(n \cdot f(n)) = \Theta(n^4)$.

    (b) The terms grow polynomially, which is "closer" to geometric than to arithmetic. The sum is then dominated by the biggest term. Hence, $\sum_{i=1}^{n} f(i) = \Theta(\text{biggest term}) = \Theta(f(n)) = \Theta(n^3)$.

    (c) Neither of these are true.

15. $f(n) = 5n^9 - 4n^3 + 7n^3 + 9$ and $g(n) = n^9$. Which line is wrong and/or in the wrong order?

    (a) $f(n) = \mathcal{O}(g(n))$ is defined as $\exists c, \exists n_0, \forall n \geq n_0, f(n) \leq cg(n)$.

    (b) Let n be really big.

    (c) Let $c = 5 + 7 + 9$ and $n_0 = 1$.

    (d) $f(n) = 5n^9 - 4n^3 + 7n^3 + 9 \leq 5n^9 + 7n^9 + 9n^9 \leq cn^9$

    (e) All right.

Counting/Infinite:

16. You get to heaven. It is beautiful with an infinite number of types of flowers. You over hear God telling the gardener to be sure to water the flower RoseyRoseRose. When you ask, you are assured that the gardener can identify each and every flower in this way. Which of the following is NOT a correct statement?

    (a) The implication is that each flower has a finite name that identifies it.

    (b) Jeff assured us that if every item in a set has a finite identifying name then the number of them is countably infinite.

    (c) The set of finite names is effectively the same as the set of finite length ASCII strings which effectively can be viewed as the set of integers which by definition is countable. If each of flowers is assigned a unique one of these then there cant be more flowers than integers.

    (d) If there were more than one flower with the same name RoseyRoseRose, then the number of flowers might be uncountable.

    (e) If there a flower had more than one name RoseyRoseRose and RoseyRoseRoseRose, then the number of flowers might be uncountable.

17. You decide to put some infinite subset of the flowers in a vase (i.e. jar). But you don't know which flowers to choose. So God gives you a different vase for each way that you could possibly choose the flowers. Which of the following is NOT a correct statement?

    (a) For you to tell me which flowers you choose, you must tell me for each flower whether or not to take it. This (at least for a random one) requires an infinite description.

    (b) Jeff assures us that this means that the number of ways to choose and hence the number of vases is uncountable.

    (c) Suppose each flower arrangement considered has a name, eg "Flowers for Jeff's 60th birthday," then the number of vases needed is finite.

    (d) The number of vases needed is the same as the number of real numbers.

    (e) All are correct.

18. You see a normal pin for sewing. On its head you see lots of angels dancing. You know that the question "How many angels can dance on the head of a pin?" has been discussed by Christians like Thomas Aquinas since the 17th century (and thought by many to be silly). Given the opportunity, you get out a ruler and measure. Sure enough, each angel is bigger than a Planck's length (1.6×10^{-35} meters) in each dimension and they don't overlap. Which of the following is NOT a correct statement?

    (a) Finite. The area of the head of the pin must be around $1mm^2 = 10^{-6}m^2$. The area an angel uses up is at least one Planck's length squared, which is $(10^{-35}m)^2$. So the total number of angels can be at most $10^{-6+2\times35} = 10^{64}$. This is big but finite.

    (b) The number of angels that can squeeze on the pin is countably infinite.

    (c) π has a finite description because there is a computer program that if given the chance would run forever printing its digits.

    (d) Becaues there are more computational problems than algorithms, some problems are uncomputable.

    (e) All true.

Structural Induction & Linear Recurrences:
Let $A_n = \{s \in \{0,1,2,3,4\}^* \mid cost(s) = n\}$ denote the set of strings of 0, 1, 2, 3, and 4 that have cost n, where the cost is 1 per character 0, 1, and 2 and is 2 per character 3 and 4. For example, $cost(02314) = 1+1+2+1+2 = 7$.
Suppose by way of strong (structural) induction, we understand the structure and size of A_i for $i < n$. We learn the structure and size of A_n as follows: Make A_n into smaller A_i which we know about. Use this information about A_i to learn about A_n.

19. Take the set A_n. Throw way all strings except those ending in the character 3. Delete the ending 3 from each remaining string. What does this do to the cost of each string? What is the set of strings you have left?

    (a) A_{n-1}

    (b) A_{n-2}

    (c) A_{n-3}

    (d) They will be missing a 3.

20. Give the recurrence relation giving the number of strings in the set A_n.

    (a) $|A_n| = |A_{n-1}| + |A_{n-2}|$

    (b) $|A_n| = |A_{n-1}| + |A_{n-2}| + |A_{n-3}| + |A_{n-4}|$

    (c) $|A_n| = 3|A_{n-1}| + 2|A_{n-2}|$

    (d) $|A_n| = 2|A_{n-1}| + 3|A_{n-2}|$

    (e) None of these

21. Let's work backwards. The quadratic $(r-3)(r+2) = r^2 - r - 6$ has roots $r_1 = 3$ and $r_1 = -2$. Give the recurrence relation giving a_n.

    (a) $a_n = 3a_{n-1} - 2a_{n-2}$

    (b) $a_n = -2a_{n-1} + 3a_{n-2}$

    (c) $a_n = a_{n-1} + 6a_{n-2}$

    (d) $a_n = -a_{n-1} - 6a_{n-2}$

    (e) None of these

22. Suppose $a_0 = 2$ and $a_1 = 16$. Give the closed form equation for a_n.

    (a) $a_n = 4 \cdot 3^n - 2 \cdot (-2)^n$

(b) $a_n = 6 \cdot 3^n - 4 \cdot (-2)^n$

(c) $a_n = -2.4 \cdot (-3)^n + 4.4 \cdot 2^n$

(d) $a_n = 2 \cdot 3^n + 16 \cdot (-2)^n$

(e) $a_n = 4 \cdot 2^n - 2 \cdot 16^n$