EECS 1019 – Practice 1

Instructor: Jeff Edmonds

Not to be handed in.

1. Multiple Choice. Which sentence relates best to the given English?

   (a) Lumber, together with marijuana, are big exports: a) $p \wedge q$; b) $p \vee q$; c) $p \oplus q$; d) other

   - Answer: a

   (b) x: a) T/F variable; b) T/F sentence; c) object; d) other

   - Answer: c

   (c) α: a) T/F variable; b) T/F sentence; c) object; d) other

   - Answer: b

   (d) Contrapositive of $p \rightarrow q$: a) $\neg p \rightarrow \neg q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; d) a & b; e) other

   - Answer: c

   (e) p is sufficient for q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   - Answer: a

   (f) p is great with q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   - Answer: e

   (g) p only if q. I read this one as a threat "You can have desert only if you eat your spinach." Which answer feels the most like this threat? a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   - Answer: c

   (h) Some: a) $\forall x$ I-can-have(x); b) $\exists x$ I-can-have(x); c) other

   - Answer: b

   (i) Can I have any?: a) $\forall x$ I-can-have(x); b) $\exists x$ I-can-have(x); c) other

   - Answer: b

   (j) Everyone is married: a) $\forall x\ \exists y\ loves(x, y)$; b) $\exists y\ \forall x\ loves(x, y)$; c) other

   - Answer: a

   (k) Every real has an inverse: a) $\forall x\ \exists y$; b) $\exists y\ \forall x$; c) other

   - Answer: a

   (l) Every Christian has God: a) $\forall x\ \exists y$; b) $\exists y\ \forall x$; c) other

   - Answer: b

   (m) Jeff is not alone in that: a) $would(\text{Jeff}) \wedge \forall x \neq \text{Jeff}\ \neg would(x)$;
   b) $would(\text{Jeff}) \wedge \exists x \neq \text{Jeff}\ \neg would(x)$; c) $would(\text{Jeff}) \wedge \exists x \neq \text{Jeff}\ would(x)$; d) other

   - Answer: c

2. The goal is to translate any truth table for a Boolean formula/sentence into Disjunctive Normal Form (DNF).
   Such a sentence is the $\vee$/OR of many clauses.
   Each such clause is the $\wedge$/AND of many literals.
   Each such literal is either a variable or its negation.
   Eg. $(A \wedge \neg B \wedge \neg C) \vee (B \wedge E \wedge F)$.

   (a) Each row of the truth table, gives the evaluation of the sentence under a given an assignment A. Such an assignment gives T/F value to each of the variables. Construct a clause that says "The variables have assignment A". Denote this clause with $clause(A)$. For example, what would clause $clause(A)$ be that is equivalent to stating the assignment is $A = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = T) \rangle$?

   - Answer: One says $p_1 = T$ with the literal p_1. One says $p_2 = F$ with the literal $\neg p_2$. We need all of these to be true. Hence, the equivalent clause is $clause(A) = (p_1 \wedge \neg p_2 \wedge p_3)$.

(b) Given a truth table for sentence S, let $S_T = \{A \mid A \text{ satisfies } S\}$ be the set of assignments A under which formula S evaluates to be true, i.e., the assignment could be A_1 OR A_2 OR .... Here each satisfying assignments in S_T is listed. For example, $S = p_1 \oplus p_2$ is satisfied iff exactly one of the variables is true, i.e., the assignment is $A_1 = \langle (p_1 = T) \wedge (p_2 = F) \rangle$ or is $A_2 = \langle (p_1 = F) \wedge (p_2 = T) \rangle$. Explain how to form a DNF expression for a general sentence S. For example, what is it for the specific sentence $S = p_1 \oplus p_2$.

- Answer: The answer is the OR of these clauses, because the expression is true iff the assignment is one of these. The DNF will be $clause(A_1) \vee clause(A_2) \vee \ldots$. Here each satisfying assignments in S_T is listed. For example $(p_1 \oplus p_2) \equiv [(p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2)]$.

(c) Consider $p_1 \oplus p_2 \oplus p_3 \oplus \ldots \oplus p_n$. For which of the T/F assignments is this true? What is this sentence called?

- Answer: When an odd number of the variables are true. It is called Parity.

(d) How many clauses would its full DNF have?

- Answer: There are 2^n possible assignments to n variables. Half of them satisfy parity. Hence, there are $\frac{1}{2}2^n$ such clauses.

(e) Consider the equivalence $(\alpha \wedge p) \vee (\alpha \wedge \neg p) \equiv \alpha$. It collapses the two clauses into one with the variable p removed. Note how if α is satisfied, then the variable p can flip between T and F. Use the rules in the purple table to prove the

- Answer: By the distributive rule we can factor out the α,
namely $(\alpha \wedge p) \vee (\alpha \wedge \neg p) \equiv \alpha \wedge (p \vee \neg p)$.
By excluded middle, $(p \vee \neg p) \equiv T$.
This gives $\alpha \vee T$, which can be simplified to α.

(f) Suppose there are two satisfying assignments/clauses that are the same for all variables, except the value of one of the variables is flipped. For example, suppose the sentence S is satisfied with both the assignment $A_1 = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = T) \rangle$ and $A_2 = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = F) \rangle$. How can you use the previous question to collapse/merge these into one equivalent clause?

- Answer: We merge the two clauses by keeping the partial assignment and dropping the variable that can have either value. $clause(A_1) = (p_1 \wedge \neg p_2 \wedge p_3)$, $clause(A_2) = (p_1 \wedge \neg p_2 \wedge \neg p_3)$, and $clause(A_1 \vee A_2) = (p_1 \wedge \neg p_2)$.

(g) Suppose the sentence S is $p_1 \vee p_2$. It has three satisfying assignments, $A_1 = \langle (p_1 = T) \wedge (p_2 = T) \rangle$, $A_2 = \langle (p_1 = T) \wedge (p_2 = F) \rangle$, and $A_3 = \langle (p_1 = F) \wedge (p_2 = T) \rangle$. What do these clauses merge into? Hint: one clause can merge with more than one other clause.

- Answer: $clause(A_1) = (p_1 \wedge p_2)$ and $clause(A_2) = (p_1 \wedge \neg p_2)$ collapse into simply p_1. $clause(A_1) = (p_1 \wedge p_2)$ and $clause(A_3) = (\neg p_1 \wedge p_2)$ collapse into simply p_2. The resulting sentence is the OR of the resulting clauses, namely S is $p_1 \vee p_2$. This is what we started with.

(h) Suppose you have an assignment A that satisfies sentence $S = p_1 \oplus p_2 \oplus p_3 \oplus \ldots \oplus p_n$. If you keep all the variables fixed except for one, and flip the value of the remaining variable, what happens to the resulting value of S? Can any of the clauses of S collapse?

- Answer: Being Parity, if A satisfies S then an odd number of its variables are true. If you flip one value, the number true will flip from odd to even making S false. This demonstrates that no two clause can collapse. S needs to retain all $\frac{1}{2}2^n$ of its clauses.

3. Proofs using Purple table:

(a) Your mother insists that you either put out the garbage or do the dishes. You convince her that you have put out the garbage and run out the door. State the rule used and its name (in purple table or in book)

- Answer: Building/Eval Or: From α, conclude $\alpha \vee \gamma$.
  Reason NOT NEEDED: It is true because $\vee$ means that at least one of these is true. If we already know that α is true, then we are done.
  From *garbage*, build [*garbage* $\vee$ *dishes*].

(b) If you put out the garbage, your mother will be happy. If you do the dishes, your mother will be happy. Prove that if (you put out the garbage or you do the dishes), your mother will be happy. Prove this about *garbage*, *dishes* and *happy*.
Hint: My proof uses two rules in the purple table and 11 lines.

- Answer:

1) *garbage* $\rightarrow$ *happy*	Axiom
2) *dishes* $\rightarrow$ *happy*	Axiom
3) Deduction Goal: (*garbage* $\vee$ *dishes*) $\rightarrow$ *happy*.	
4) *garbage* $\vee$ *dishes*	Assumption/Premise
5) Cases Goal: *happy*. Cases *garbage* and *dishes*.	
6) Case 1: *garbage*	Assumption/Premise
7) *happy*	Modus Ponens 1 & 6 .
8) Case 2: *dishes*	Assumption/Premise
9) *happy*	Modus Ponens 2 & 8.
10) *happy*	Conclude cases 4, 7, & 9
11) (*garbage* $\vee$ *dishes*) $\rightarrow$ *happy*	Conclude deduction.

4. Let A denote the sentence $\exists x \forall y P(x,y)$ and let B denote $\exists y \forall x \neg P(x,y)$.
Our goal is to either construct a P for which for which both are true,
or to prove that for every evaluation of the relation P, they can't both be true.

(a) As one does for "proof by contradiction", lets start by assuming that A, i.e., $\exists x \forall y P(x,y)$ is true.
Let's determine what this says about P.
Assume that you have an oracle A that assures you that it is true.
Remember Jeff's oracle game.
- What are you allowed to give her? What then does she assure?
- What will she give you? What then does she assure?
Use the notation $c_{\langle A,\forall\rangle}$ and $c_{\langle A,\exists\rangle}$.

- Answer: With $\exists x \forall y P(x,y)$, she is assuring $\exists x$. Hence, she gives you such an object, which we will denote $x_{\langle A,\exists\rangle}$ for which she assures $\forall y P(x_{\langle A,\exists\rangle}, y)$. Now she is assuring $\forall y$. Hence, you can give her any object, which we will denote $y_{\langle A,\forall\rangle}$ for which she assures $P(x_{\langle A,\exists\rangle}, y_{\langle A,\forall\rangle})$.

(b) Fill in as much of the first table that you know.
In the first row and column, put in the names of the objects $c_{\langle A,\forall\rangle}/c_{\langle A,\exists\rangle}$ that you know.

- Answer:

A	x_1		$x_{\langle A,\exists\rangle}$	
y_1			T	
			T	
$y_{\langle A,\forall\rangle} = y_{\langle B,\exists\rangle}$			T	
			T	

B	x_1		$x_{\langle B,\forall\rangle} = x_{\langle A,\exists\rangle}$	
y_1				
$y_{\langle B,\exists\rangle}$	F	F	F	F

(c) Now assume that you have an oracle B that assures you that $\exists y \forall x \neg P(x,y)$ is true.
Repeat the previous two questions. Use the second table this time.

- Answer: With $\exists y \forall x \neg P(x,y)$, she is assuring $\exists y$. Hence, she gives you such an object, which we will denote $y_{\langle B,\exists\rangle}$ for which she assures $\forall x \neg P(x, y_{\langle B,\exists\rangle})$. Now she is assuring $\forall x$. Hence, you can give her any object, which we will denote $x_{\langle B,\forall\rangle}$ for which she assures $\neg P(x_{\langle B,\forall\rangle}, y_{\langle B,\exists\rangle})$.

(d) Can both of these statements be true at the same time? Yes or No?
If Yes, your tables should be showing such an example.
If No, reveal to us a contradiction.
Use the oracles again. Give and receive objects from them, until oracle A assures us of some fact and oracle B assures us of a different fact and these contradict each other, i.e., $\beta \wedge \neg\beta$.

- Answer: No. They cannot both be true.
By way of contradiction assume that both are true.
We begin with the first half of each of the previous games.
Oracle A gives you $x_{\langle A,\exists\rangle}$ and Oracle B gives you $y_{\langle B,\exists\rangle}$.
Oracle A is assuring that $\forall y P(x_{\langle A,\exists\rangle}, y)$. Hence, you can give her any object $y_{\langle A,\forall\rangle}$. You give her $y_{\langle B,\exists\rangle}$. She then assures $P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$.
Oracle B is assuring that $\forall x \neg P(x, y_{\langle B,\exists\rangle})$. Hence, you can give her any object $x_{\langle B,\forall\rangle}$. You give her $x_{\langle A,\exists\rangle}$. She then assures $\neg P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$.
These $P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$ and $\neg P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$ contradict each other.
Hence, the sentence $[\exists x \forall y P(x,y)] \wedge [\exists y \forall x \neg P(x,y)]$ can never be true.

(e) Redo your previous proof more formally without mention of oracles.

- Answer: No. They cannot both be true.

0) Proof by contradiction goal: $\neg(A \wedge B)$

1)	$A \wedge B$	assumption/premise
2)	$\exists x \forall y P(x,y)$	Separating And (1)
3)	$\exists y \forall x \neg P(x,y)$	
4)	$\forall y P(x_{\langle A,\exists\rangle}, y)$	Names the value of x claimed to exist in (2)
5)	$\forall x \neg P(x, y_{\langle B,\exists\rangle})$	Names the value of y claimed to exist in (3)
6)	$P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle}))$	(4) true forall y so true for this one
7)	$\neg P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$	(5) true forall x so true for this one
8)	$P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})) \wedge \neg P(x_{\langle A,\exists\rangle}, y_{\langle B,\exists\rangle})$	Build And (6) & (7)
9)	Contradiction	Excluded Middle (8)

10) By way of contradiction, they cannot both be true.