EECS/MATH 1019: Discrete Mathematics for Computer Science

Section C — Prof. N. Madras

**Solutions to First Midterm Test — Version A** — October 18, 2023

1. [3 marks] Let $S(x)$ be the predicate "$x$ is a student," let $P(y)$ be the predicate "$y$ is a professor," and let $A(x,y)$ be the predicate "$x$ has asked $y$ a question." The domain is the set of all people. Use quantifiers to express the statement

> "There is a professor who has never been asked a question by a student."

Do not use set notation (such as "$\in$") in your answer.

**Solution:** Here are two correct ways to express the statement.

$$\begin{array}{ll} (1) & \exists y\ (P(y) \wedge \neg\exists x\, (S(x) \wedge A(x,y))) \\ (2) & \exists y\, (P(y) \wedge \forall x\, \neg(S(x) \wedge A(x,y)))\,. \end{array}$$

2. [5 marks] Use rules of inference and logical equivalences to show that the conclusion "It was cold this morning" follows from the premises "If it was not raining this morning or if it was not cold this morning, then Erin will ride her bike to school" and "Erin did not ride her bike to school." You may use the tables given near the beginning of this test paper. Give a detailed formal argument.

**Solution:** Let $C$ be the statement "It was cold this morning."
Let $R$ be the statement "It was raining this morning."
Let $B$ be the statement "Erin rode her bike to school."

Here is a formal argument.

| | | | |
|---|---|---|---|
| (1) | | $(\neg R \vee \neg C) \to B$ | First premise |
| (2) | | $\neg B$ | Second premise |
| (3) | $\therefore$ | $\neg(\neg R \vee \neg C)$ | Modus tollens from (1) and (2) |
| (4) | $\therefore$ | $R \wedge C$ | De Morgan's Law from (3) |
| (5) | $\therefore$ | $R$ | Simplification from (4) |

Here is a different version, which is also valid.

| | | | |
|---|---|---|---|
| (1) | | $(\neg R \vee \neg C) \to B$ | First premise |
| (2) | $\therefore$ | $\neg B \to \neg(\neg R \vee \neg C)$ | Contrapositive of (1) |
| (3) | | $\neg B$ | Second premise |
| (4) | $\therefore$ | $\neg(\neg R \vee \neg C)$ | Modus ponens from (2) and (3) |
| (5) | $\therefore$ | $R \wedge C$ | De Morgan's Law from (4) |
| (6) | $\therefore$ | $R$ | Simplification from (5) |

3. [4 marks] Write an expression equivalent to the following that uses only standard logic notation but does not use the negation symbol $\neg$.

$$\neg\forall y\,(Q(y) \wedge \exists x(\neg R(x,y))\,)$$

(*Suggestion:* You may wish to use the symbol $\to$.)

**Solution:**

$$\begin{aligned}
& \neg\,\forall y\,(Q(y) \wedge \exists x(\neg R(x,y))\,) \\
\equiv\ & \exists y\,\neg\,(Q(y) \wedge \exists x(\neg R(x,y))\,) \\
\equiv\ & \exists y\,(\neg Q(y) \vee \neg\,\exists x(\neg R(x,y))\,) \qquad \text{(De Morgan's Law)} \\
\equiv\ & \exists y\,(\neg Q(y) \vee \forall x(\neg\neg R(x,y))\,) \\
\equiv\ & \exists y\,(\neg Q(y) \vee \forall x(R(x,y))\,) \\
\equiv\ & \exists y\,(Q(y) \to \forall x(R(x,y))\,) \qquad (\text{since } \neg p \vee q \equiv p \to q).
\end{aligned}$$

4. [7 marks] Prove that if $n$ is an odd integer, then there exists a unique integer $k$ such that $n$ is the sum of $k$ and $k+5$.

(Note that you need to prove both existence and uniqueness.)

**Solution:** (*Comment: We start with some observations and calculations to help us figure out what is going on. Recall that if $n$ be an odd integer, then $n = 2m+1$ for some integer $m$. The sum of $k$ and $k+5$ is $2k+5$. Can this* *equal $2m+1$? Solving $2k+5 = 2m+1$ results in $m = k+2$, which is that same as $m-2 = k$. This will be useful, but we need to make it into a proof.*)

Here is the proof.

Let $n$ be an odd integer. Then there is an integer $m$ such that $n = 2m+1$. Let $k = m-2$. Observe that the sum of $k$ and $k+5$ is $2k+5$, which equals $2(m-2)+5$, which equals $2m-4+5$, which equals $2m+1$, which is $n$. This proves that if $n$ is an odd integer, then there exists an integer $k$ such that $n$ is the sum of $k$ and $k+5$.

Now, for a given odd integer $k$, why is there a unique such $k$? Suppose that $k_1$ and $k_2$ are two integers such that $n = k_1+(k_1+5)$ and also $n = k_2+(k_2+5)$. Then we would have $2k_1+5 = 2k_2+5$. Then it follows that $k_1 = k_2$ (in detail: by subtracting 5 from both sides, we obtain $2k_1 = 2k_2$; then, upon dividing both sides by 2, we obtain $k_1 = k_2$). That is, there cannot be two different $k$'s such that $n$ is the sum of $k$ and $k+5$. This completes the proof of uniqueness.

5. [8 marks] First, recall our usual notation: The set of natural numbers is $\mathbb{N} = \{0,1,2,3,\ldots\}$, and the set of positive integers is $\mathbb{Z}^+ = \{1,2,3,4,\ldots\}$.

Let $W$ be the set of all real numbers of the form $m+\sqrt{n}$ where $n$ is a natural number and $m$ is either 1 or 2. That is,

$$W \;=\; \left\{ m+\sqrt{n} \,\middle|\, n \in \mathbb{N} \text{ and } m \in \{1,2\} \right\}.$$

(a) Prove that $\mathbb{Z}^+ \subseteq W$.

(b) Is $W$ a countable set? Justify your answer.

**Solution:** (a) Let $x \in \mathbb{Z}^+$. Then $x-1 \in \mathbb{N}$. Now, let $m = 1$ and $n = (x-1)^2$. Then $n$ is a natural number, so $m+\sqrt{n}$ is an element of $W$ by definition. And we have

$$\begin{aligned} m+\sqrt{n} \;=\; 1+\sqrt{(x-1)^2} \;&=\; 1+(x-1) \quad (\text{since } x-1 \geq 0) \\ &=\; x\,. \end{aligned}$$

Since we know that $m+\sqrt{n} \in W$, the above equations prove that $x \in W$. We conclude that every element of $\mathbb{Z}^+$ is in $W$. Therefore $\mathbb{Z}^+ \subseteq W$.

(b) Yes, $W$ is countable. To prove this, let $W_1$ and $W_2$ be the sets

$$\begin{aligned} W_1 &= \{1+\sqrt{n} \,|\, n \in \mathbb{N}\}, \\ W_2 &= \{2+\sqrt{n} \,|\, n \in \mathbb{N}\}. \end{aligned}$$

Then $W_1$ is countable because its elements are in one-to-one correspondence with $\mathbb{N}$ by the correspondence $n \leftrightarrow 1+\sqrt{n}$. (Alternatively, it is because we can list the elements of $W_1$ as a sequence: $1+\sqrt{0}, 1+\sqrt{1}, 1+\sqrt{2}, 1+\sqrt{3}, \ldots$) Similarly, $W_2$ is countable because its elements are in one-to-one correspondence with $\mathbb{N}$ by the correspondence $n \leftrightarrow 2+\sqrt{n}$.

Finally, since $W = W_1 \cup W_2$ and the union of two countable sets is a countable set, it follows that $W$ is countable.

6. [3 marks] Let the universal set be $U = \{5, 6, 7\}$. Let $A = \{5, 7\}$ and $B = \{7\}$.

Write the elements of the set $\overline{A} \times \overline{B}$, the Cartesian product of the complements of $A$ and of $B$.

**Solution:** (a) $\overline{A} = \{6\}$ and $\overline{B} = \{5, 6\}$, so $\overline{A} \times \overline{B} = \{(6, 5), (6, 6)\}$.

7. [3 marks] Evaluate

$$\sum_{k=3}^{5} \left\lfloor \frac{k}{2} \right\rfloor .$$

(Note the use of the floor function in the above expression.)

**Solution:** This sum equals

$$\left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{5}{2} \right\rfloor, \quad \text{which equals} \quad 1 + 2 + 2 \quad = \quad 5.$$

8. [7 marks] Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by

$$f(m, n) \;=\; (m + n, n)\,.$$

(Recall that $\mathbb{N}$ is the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$.)
For example, the image of the ordered pair $(3, 2)$ is $(5, 2)$.

(a) Find a preimage of $(100, 70)$.

(b) Is $f$ a surjection (that is, is $f$ onto)? Explain.

**Solution:**
(a) A preimage is $(30, 70)$, since $f(30, 70) \;=\; (30 + 70, 70) \;=\; (100, 70)$.

(b) No. For example, $(1,2)$ is in the codomain, but we shall prove by contradiction that it is not in the range. Assume $f(m,n) = (1,2)$ for some $(m,n) \in \mathbb{N} \times \mathbb{N}$. Then $m + n = 1$ and $n = 2$. It follows that $m = -1$. So $(m,n) = (-1,2)$, which is not in the domain $\mathbb{N} \times \mathbb{N}$.