Solutions to Final Exam for EECS/MATH 1019, April 2023

1. (3 marks) Decide which of the following is logically equivalent to the statement "If a graph is connected, then it either has an Euler circuit or it has at least one vertex with odd degree." (For this question, no explanation is needed.)

(a) If a graph has at least one vertex of odd degree, then if it is connected it must have an Euler circuit.

(b) If all of the vertices of a graph have even degree, then the graph is connected and it has an Euler circuit.

(c) If all of the vertices of a graph have even degree, then if it is connected it must have an Euler circuit.

(d) If a graph has an Euler circuit and all of its vertices have even degree, then the graph is connected.

(e) None of the above.

Answer: (c).
(*Remark:* Statements (a) and (b) are not true for all graphs; statement (d) is true for all graphs, but it is not equivalent to the given statement.)

2. (4 marks) Calculate

$$\sum_{i=1}^{3}\sum_{j=2}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right)\left(\frac{j(j+1)}{2}-\frac{j(j-1)}{2}\right)$$

(*Suggestion*: Do some algebraic manipulation before plugging in numbers.)

Solution: First observe that

$$\frac{j(j+1)}{2}-\frac{j(j-1)}{2} \;=\; \frac{j^2+j-(j^2-j)}{2} \;=\; \frac{j+j}{2} \;=\; j\,.$$

Then we have

$$
\begin{aligned}
& \sum_{i=1}^{3}\sum_{j=2}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right)\left(\frac{j(j+1)}{2}-\frac{j(j-1)}{2}\right) \\
&= \sum_{i=1}^{3}\sum_{j=2}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right)\times j \\
&= \sum_{i=1}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right)(2+3) \\
&= 5\sum_{i=1}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right) \\
&= 5\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}\right) \\
&= 5\left(1-\frac{1}{4}\right) \\
&= 5\times\frac{3}{4} \quad = \quad \frac{15}{4}.
\end{aligned}
$$

3. (6 marks) Prove that if x^3 is an irrational number, then x is irrational.

Solution: Since this statement is logically equivalent to its contrapositive, we shall prove the contrapositive: *If x is rational, then x^3 is rational.* (We could also give a proof by contradiction along similar lines.)

Assume that x is rational. Then $x = m/n$ for some integers m and n with $n \neq 0$. Then

$$x^3 \;=\; \left(\frac{m}{n}\right)^3 \;=\; \frac{m^3}{n^3}\,.$$

Since m^3 and n^3 are integers, with $n^3 \neq 0$, we conclude that x is rational. This proves the contrapositive.

4. (6 marks) Consider a function $h : A \to A$. Assume that h is onto. Must $h \circ h$ also be onto? Answer "YES" or "NO" and justify your answer.

Solution: YES. Let $x \in A$. Since h is onto, there is a w in A such that $h(w) = x$. And again, since h is onto, there is a v in A such that $h(v) = w$.

Then

$$(h \circ h)(v) \;=\; h(h(v)) \;=\; h(w) \;=\; x\,.$$

This shows that for every x in A, there is a v in A such that $(h \circ h)(v) = x$. This proves that $h \circ h$ is onto.

5. (5 marks) For each integer x, let

$$f(x) \;=\; \frac{x^6 + x^2 - 1}{x^2 + 3}\,.$$

Prove that $f(x)$ is $O(x^4)$. Be sure to be explicit about your choices of the witnesses C and k from the definition.

Solution: For $x \geq 1$, we have

$$|f(x)| \;=\; \frac{x^6 + x^2 - 1}{x^2 + 3} \;\leq\; \frac{x^6 + x^2 - 0}{x^2 + 3} \;\leq\; \frac{x^6 + x^6}{x^2 + 3} \;\leq\; \frac{x^6 + x^6}{x^2} \;=\; 2x^4$$

(we used the fact that $x^2 \leq x^6$ when $x \geq 1$). Therefore we have proved that $f(x) = O(x^4)$ with witnesses $C = 2$ and $k = 1$.

6. (7 marks) Use Mathematical Induction to prove that for every positive integer n,

$$\sum_{i=1}^{n} \frac{i}{2^i} \;=\; \frac{2^{n+1} - 2 - n}{2^n}\,.$$

Solution: For each positive integer n, let $P(n)$ be the statement that

$$\sum_{i=1}^{n} \frac{i}{2^i} \;=\; \frac{2^{n+1} - 2 - n}{2^n}$$

Basis step: For $n = 1$, $P(1)$ says that

$$\frac{1}{2^1} \;=\; \frac{2^2 - 2 - 1}{2^1}\,,$$

which is true because both sides of the equation equal $1/2$.

Inductive step: Now assume that k is a positive integer and that $P(k)$ is true. Then we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{2^i} &= \left(\sum_{i=1}^{k} \frac{i}{2^i}\right) + \frac{k+1}{2^{k+1}} \\
&= \frac{2^{k+1}-2-k}{2^k} + \frac{k+1}{2^{k+1}} \qquad \text{(since } P(k) \text{ is true)} \\
&= \frac{2(2^{k+1}-2-k) + (k+1)}{2^{k+1}} \\
&= \frac{2(2^{k+1}) - 4 - 2k + k + 1}{2^{k+1}} \\
&= \frac{2^{(k+1)+1} - 3 - k}{2^{k+1}} \\
&= \frac{2^{(k+1)+1} - 2 - (k+1)}{2^{k+1}}.
\end{aligned}$$

This proves that $P(k+1)$ is true.

Since we proved $P(1)$ and $\forall k \in \mathbb{Z}^+ \, (P(k) \rightarrow P(k+1))$, we conclude from the principle of mathematical induction that $P(n)$ is true for every positive integer n.

7. (16 marks = 1 + 7 + 2 + 5 + 1) Consider a recursively defined sequence with

$$c_0 = 6, \; c_1 = 2, \quad \text{and} \quad c_n = \frac{1}{2}(c_{n-1} + c_{n-2}) \; \text{ when } n \geq 2.$$

(a) Get a feel for the sequence $\{c_n\}$ by computing c_2, c_3, and c_4.
(b) Use strong induction to prove that $\forall n \in \mathbb{N} \, (0 < c_n < 8)$.
(c) The expression for c_n takes the form of a linear recurrence. What is the characteristic equation of the recurrence?
(d) Determine a formula for c_n.
(e) Does your formula agree with your computation of c_3 in part (a)?

Solution: (a) $c_2 = \frac{1}{2}(2+6) = 4$, $c_3 = \frac{1}{2}(4+2) = 3$, and $c_4 = \frac{1}{2}(3+4) = \frac{7}{2} = 3.5$.

(b) For each $n \in \mathbb{N}$, let $P(n)$ be the statement $0 < c_n < 8$.

Basis step: Since $c_0 = 6$ and and $c_1 = 2$, we see that $P(0)$ and $P(1)$ are true.

Inductive step: Assume $k \geq 1$ and that $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ is true. Then in particular $P(k)$ and $P(k-1)$ are both true, so $0 < c_k < 8$ and $0 < c_{k-1} < 8$. Then we have

$$c_{k+1} \;=\; \frac{1}{2}(c_k + c_{k-1}) \;>\; \frac{1}{2}(0+0) \;=\; 0$$

and

$$c_{k+1} \;=\; \frac{1}{2}(c_k + c_{k-1}) \;<\; \frac{1}{2}(8+8) \;=\; 8\,.$$

This shows that $0 < c_{k+1} < 8$; that is, $P(k+1)$ is true. This completes the inductive step.

Having done the basis step and the inductive step, we conclude that $P(n)$ is true for every natural number n.

(c) $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$. (This is equivalent to $2r^2 - r - 1 = 0$.)

(d) Since $2r^2 - r - 1 = (2r+1)(r-1)$, the characteristic roots are $r = 1$ and $r = -1/2$. Therefore c_n must have the form

$$c_n \;=\; b\left(-\frac{1}{2}\right)^n + d(1)^n \qquad \text{for some real numbers } b \text{ and } d.$$

For $n = 0$ and $n = 1$, we obtain

$$6 \;=\; c_0 \;=\; b\left(-\frac{1}{2}\right)^0 \;+\; d(1)^0, \qquad \text{i.e.} \qquad 6 \;=\; b \;+\; d$$

$$\text{and} \quad 2 \;=\; c_1 \;=\; b\left(-\frac{1}{2}\right)^1 \;+\; d(1)^1, \qquad \text{i.e.} \qquad 2 \;=\; -\frac{b}{2} \;+\; d$$

Subtracting the first equation from the second gives

$$6 - 2 \;=\; (b+d) - \left(-\frac{b}{2} + d\right) \qquad \therefore\; 4 \;=\; \frac{3b}{2} \qquad \therefore\; b \;=\; \frac{8}{3}$$

and hence $d \;=\; 6 - b \;=\; 6 - \frac{8}{3} \;=\; \frac{18}{3} - \frac{8}{3} \;=\; \frac{10}{3}$. So the solution is

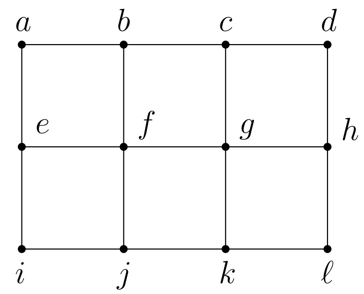
$$c_n \;=\; \frac{8}{3}\left(-\frac{1}{2}\right)^n \;+\; \frac{10}{3}\,.$$

(e) From the formula of (d), we find that

$$c_3 \;=\; \frac{8}{3}\left(-\frac{1}{2}\right)^3 + \frac{10}{3} \;=\; \frac{8}{3}\left(-\frac{1}{8}\right) + \frac{10}{3} \;=\; -\frac{1}{3} + \frac{10}{3} \;=\; \frac{9}{3} \;=\; 3,$$

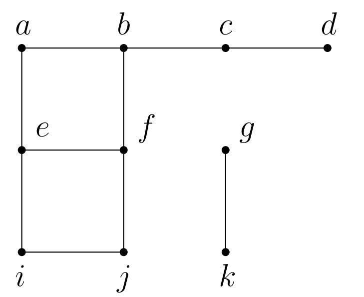
which agrees with the calculation from part (a).

8. (4 marks) Let G be the following graph:



Find a subgraph H of G that has all of the following properties:
H has 10 vertices and 10 edges, H is disconnected, and there is a path from vertex d to vertex i in H.

Solution: Here is one example:



9. (6 marks) Assume that R is an equivalence relation on the nonempty set A, and assume that

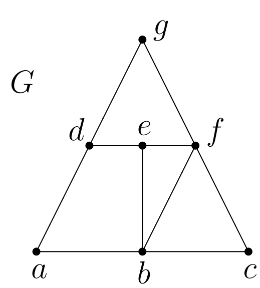
$$\forall x\,\forall y\,\exists z\,(\,(x,z)\notin R \text{ and } (y,z)\notin R\,).$$

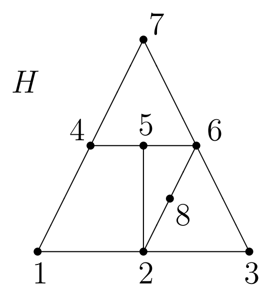
Prove that R has at least three equivalence classes.

Solution: First, could there be only one equivalence class? To show that this is not possible, let $x \in A$. Whatever y is, we see that there exists a z such that $(x, z) \notin R$. That is, x and z are in different equivalence classes. So there are at least two equivalence classes.

Let x be in one equivalence class, and let y be in another equivalence class. Then there exists a z that is not related to x and is not related to y. So z is not in $[x]$ (the equivalence class of x) nor is z in $[y]$ (the equivalence class of y). Therefore the equivalence class $[z]$ is different from both $[x]$ and $[y]$. This shows that there are at least three equivalence classes.

10. (8 marks) Prove that one of these graphs is bipartite, and explain how you know that the other graph is not bipartite.

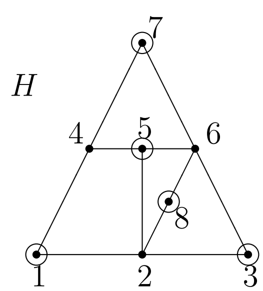




Solution: The graph H is bipartite, as is demonstrated by the following bipartition, with vertices of V_1 circled in the diagram:

$V_1 = \{1, 3, 5, 7, 8\}$

$V_2 = \{2, 4, 6\}$



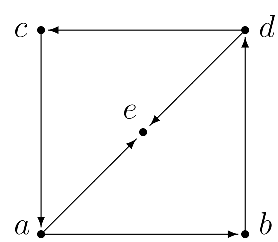
Why is G not bipartite? Suppose that V_1 and V_2 formed a bipartition of G. Without loss of generality, assume $b \in V_1$. Since e and f are neighbours of b, they must both be in V_2. But the edge (e, f) is in G, and both of its endpoints are in V_2, so this cannot be a bipartition. This contradiction proves that G is not bipartite.

Alternatively, G has a cycle of length 3 ($befb$), and it was observed in the text and in class that a graph cannot be bipartite if it contains a circuit of odd length.

11. (15 marks $= 2 + 2 + 4 + 3 + 4$) Here is a new definition:

> **Definition:** Let G be a directed graph, and let v and w be vertices of G. We say that v and w are *mutually reachable* if there is a path from v to w and a path from w to v. (Remember that a path of length 0 starts and ends at the same vertex.)

Now let H be the following directed graph:



(a) Show that a and c are mutually reachable in H.

(b) Show that d and e are not mutually reachable in H.

(c) For an arbitrary directed graph G, the relation "mutually reachable" on the set of vertices is evidently symmetric (from the definition) as well as reflexive (because of paths of length 0). Show that the relation "mutually reachable" is transitive.

(d) Part (c) shows that "mutually reachable" is an equivalence relation. The equivalence classes of the "mutually reachable" relation are the vertex sets of the strongly connected components. (You do not need to justify the preceding statements.) List the vertex sets of the strongly connected components of H.

(e) We say that a directed edge (x, y) is *flippable* if replacing (x, y) by the new directed edge (y, x) makes the directed graph strongly connected. Does H have any flippable edges? Explain.

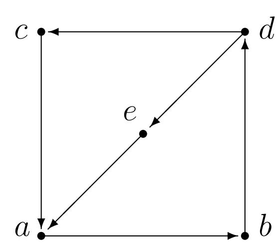
Solution: (a) We see that $abdc$ is a path from a to c, and that ca is a path from c to a in this directed graph. Therefore a and c are mutually reachable.

(b) There is no path from e to any other vertex, since H has no edge that starts at e. Therefore d and e are not mutually reachable. (There is a path from d to e, but that is not enough.)

(c) Let v, w, and x be vertices such that v and w are mutually reachable and also w and x are mutually reachable. Then there is a path from v to w and a path from w to x. Joining the beginning of the second path to the end of the first path (at the vertex w) produces a path in H from v to x. Similarly, there is a path from x to w and a path from w to v. Joining the beginning of the second path to the end of the first path (at the vertex w) produces a path in H from x to v. We conclude that v and x are mutually reachable. This shows that the relation is transitive.

(d) Observe that each of a, b, c, and d is mutually reachable from any of these four vertices, but e is not mutually reachable from any vertex. Therefore H has two strongly connected components: one has vertex set $\{a, b, c, d\}$, and the other consists of the single vertex $\{e\}$.

(e) Yes. If we replace the edge (a, e) by (e, a), then we get the new digraph



which is strongly connected. Therefore (a, e) is flippable.

Simliarly, the edge (d, e) is flippable in the original digraph H.