York University

Faculty of Science / Lassonde School of Engineering

EECS/MATH 1019 3.00

Final Examination — April 24, 2023

NAME (print):_____
(First) (Last)

SIGNATURE:_____

STUDENT NUMBER:_____

E-MAIL:_____

Please circle your section:	Section M	Prof. I. Raguimov	Tue. Thur. 10:00 a.m.
	Section N	Prof. N. Madras	Tues. Thurs. 11:30 a.m.
	Section O	Prof. M. La Croix	Tue. Thur. 5:00 p.m.

Instructions:

1. Time allowed: 180 minutes. Total score: 80 marks.
2. This exam contains 11 questions on 13 sheets of paper. **Please do not remove any pages from this exam.** You will receive a blank sheet of paper that you can use for rough work that will not be graded.
3. Students in Section O (Prof. La Croix) may use a non-programmable calculator. Except for this, no calculators, books, or other aids are permitted.
4. There are some empty pages at the end for rough work or continuation of answers. If your answer to a given question continues on another page, then clearly indicate this.
5. Explain your answers clearly. Show your work.

1. (3 marks) Decide which of the following is logically equivalent to the statement "If a graph is connected, then it either has an Euler circuit or it has at least one vertex with odd degree." (For this question, no explanation is needed.)

   (a) If a graph has at least one vertex of odd degree, then if it is connected it must have an Euler circuit.

   (b) If all of the vertices of a graph have even degree, then the graph is connected and it has an Euler circuit.

   (c) If all of the vertices of a graph have even degree, then if it is connected it must have an Euler circuit.

   (d) If a graph has an Euler circuit and all of its vertices have even degree, then the graph is connected.

   (e) None of the above.

2. (4 marks) Calculate

$$\sum_{i=1}^{3}\sum_{j=2}^{3}\left(\frac{1}{i}-\frac{1}{i+1}\right)\left(\frac{j(j+1)}{2}-\frac{j(j-1)}{2}\right)$$

(*Suggestion*: Do some algebraic manipulation before plugging in numbers.)

3. (6 marks) Prove that if x^3 is an irrational number, then x is irrational.

Continuation of answer

4. (6 marks) Consider a function $h : A \to A$. Assume that h is onto. Must $h \circ h$ also be onto? Answer "YES" or "NO" and justify your answer.

Continuation of answer

5. (5 marks) For each integer x, let

$$f(x) \;=\; \frac{x^6 + x^2 - 1}{x^2 + 3}\,.$$

Prove that $f(x)$ is $O(x^4)$. Be sure to be explicit about your choices of the witnesses C and k from the definition.

Continuation of answer

6. (7 marks) Use Mathematical Induction to prove that for every positive integer n,

$$\sum_{i=1}^{n} \frac{i}{2^i} = \frac{2^{n+1} - 2 - n}{2^n}.$$

Continuation of answer

7. (16 marks $= 1 + 7 + 2 + 5 + 1$) Consider a recursively defined sequence with

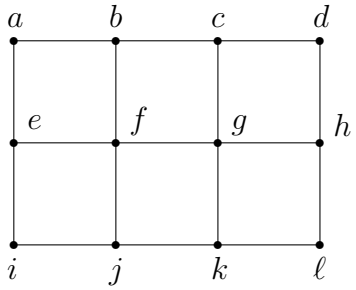
$$c_0 = 6,\ c_1 = 2, \quad \text{and} \quad c_n = \frac{1}{2}(c_{n-1} + c_{n-2}) \ \text{ when } n \geq 2.$$

(a) Get a feel for the sequence $\{c_n\}$ by computing c_2, c_3, and c_4.
(b) Use strong induction to prove that $\forall n \in \mathbb{N}\,(0 < c_n < 8)$.
(c) The expression for c_n takes the form of a linear recurrence. What is the characteristic equation of the recurrence?
(d) Determine a formula for c_n.
(e) Does your formula agree with your computation of c_3 in part (a)?

Continuation of answer

Continuation of answer

8. (4 marks) Let G be the following graph:



Find a subgraph H of G that has all of the following properties:
H has 10 vertices and 10 edges, H is disconnected, and there is a path from vertex d to vertex i in H.

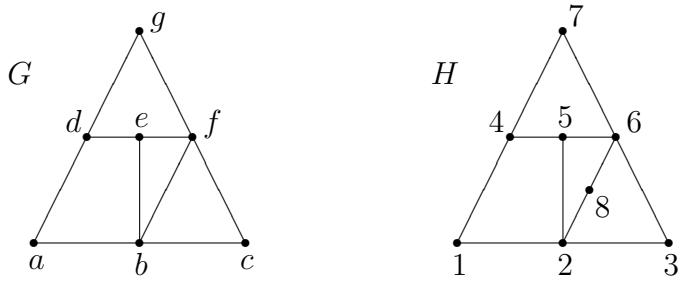
9. (6 marks) Assume that R is an equivalence relation on the nonempty set A, and assume that

$$\forall x \, \forall y \, \exists z \, (\, (x, z) \notin R \text{ and } (y, z) \notin R \,).$$

Prove that R has at least three equivalence classes.

Continuation of answer

10. (8 marks) Prove that one of these graphs is bipartite, and explain how you know that the other graph is not bipartite.

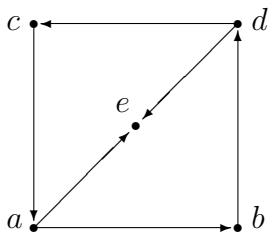


Continuation of answer

11. (15 marks $= 2 + 2 + 4 + 3 + 4$) Here is a new definition:

> **Definition:** Let G be a directed graph, and let v and w be vertices of G. We say that v and w are *mutually reachable* if there is a path from v to w and a path from w to v. (Remember that a path of length 0 starts and ends at the same vertex.)

Now let H be the following directed graph:



(a) Show that a and c are mutually reachable in H.

(b) Show that d and e are not mutually reachable in H.

(c) For an arbitrary directed graph G, the relation "mutually reachable" on the set of vertices is evidently symmetric (from the definition) as well as reflexive (because of paths of length 0). Show that the relation "mutually reachable" is transitive.

(d) Part (c) shows that "mutually reachable" is an equivalence relation. The equivalence classes of the "mutually reachable" relation are the vertex sets of the strongly connected components. (You do not need to justify the preceding statements.) List the vertex sets of the strongly connected components of H.

(e) We say that a directed edge (x, y) is *flippable* if replacing (x, y) by the new directed edge (y, x) makes the directed graph strongly connected. Does H have any flippable edges? Explain.

Continuation of answer

Continuation of answer

Extra page for continuation or rough work

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