Locker Room Problem

Problem: There are n players, each with a locker and a driver's license. The coach randomly permutes the licenses and puts one in each locker. The players can agree on a strategy. Each player independently goes into the locker room and can look in half the lockers. We say that he succeed if he finds his own license. We say that they succeed if each player succeeds to finds his own license. They are not allowed to change the room set up or communicate in any way. The probability that a given player succeeds is $\frac{1}{2}$. If things were completely independent then the probability that all succeeds would be $\frac{1}{2^n}$. Is it possible for the players to have a strategy in which they all succeed with a significantly higher probability, say 0.3?

Strategy: Each player starts by looking in his own locker. If he finds Bob's license, he looks in Bob's locker. If in Bob's locker he finds John's license, he looks in John's locker next. This continues until either he finds his locker or has looked in half the lockers.

Permutation Graph: Put a directed edge from i to j if the locker i contains license j. Having out-degree one and in-degree one, this graph contains a collection of cycles.

Success: Player i starts at node i, i.e. his own locker, and follows the edges of this graph. He succeeds when he finds his own driver's license, i.e. when the cycle he is following points back to node i, i.e. he arrives back at node i. Hence, he succeeds when the cycle that he is in contains at most half the nodes. They all succeed if the permutation graph contains no cycles of length greater than half.

Probability of a k Cycle: Let $k \in [\frac{n}{2}+1, n]$. We will show that the probability that a random permutation graph contains a k cycle is $\frac{1}{k}$.

The number of permutation graphs is $n!$ because it can be described by a permutation. There are n choices for a neighbor for node 1 and then $n-1$ choices for a neighbor for node 2, because they can't have the same neighbor, and so on.

Now let us count the number permutations with a cycle of length k. Choose a start node i_1. There are n ways. Choose its neighbor i_2. There are $n-1$ ways, because we don't want to allow node i_1. Choose i_2's neighbor i_3. There are $n-2$ ways, because we don't want to allow nodes i_1 or i_2. Continue until you choose i_{k-1}'s neighbor i_k. There are $n-(k-1)$ ways. Because we want a cycle of length k, we know that i_k's neighbor is node i_1. Then there $(n-k)!$ ways to arranging the remaining $n-k$ players. The total number of ways is $n!$. However, we over counted by a factor of k because it does not matter which of the k nodes in the k cycle that we started with. Note that we would have over counted further if there was a second cycle of length k in the remaining $n-k$ nodes, but this is not possible because $n-k < k$. Hence, the total number of permutation graphs with a cycle of length k is $\frac{n!}{k}$. The fact that the probability is $\frac{1}{k}$ follows.

Probability of a Large Cycle: There can't be two cycles of more than half the nodes. Hence, the event of there being a k cycle is disjoint for the different $k \in [\frac{n}{2}+1, n]$. Hence the probability of there being a more than half cycle is $\sum_{k=\frac{n}{2}+1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\frac{n}{2}} \frac{1}{k} \approx \ln(n) - \ln(\frac{n}{2}) = \ln(2)$. Hence, the probability of no such large cycle and hence of success is $1-\ln(2) > 0.3$.