One-per-clause NPComplete

0) In NP

1) What to reduce to

P_{easy} P_{hard}

$3\text{-Col} \le_p \text{one-per-clause}$

2) Define probs

3-Col - Instance : graph G

Solution : 3-colouring

One-per-clause

Instance : clauses of at most 3 variables

no negatives

solution : A T/F assignment

Each clause has exactly one true

Direction

3) Given oracle for one-per

Design oracle for 3-col

4) similarities variables → {T,F}, nodes → {r, b, g} ⇒ {u,r,g} → {T,F}

5) Given instance G for 3-col give instance for one-per

InstanceMap (G)

∀ node u + colour $c \in \{r, b, g\}$ ⇒ variable u_c

∀ edge {u,v} + colour c ⇒ variable $e_{\{u,v\},c}$

∀ node u ⇒ clause u_r, u_b, u_g

∀ edge {u,v} ⇒ 3 clauses $u_r, v_r, e_{\{u,v\},r}$

$u_g, v_g, e_{\{u,v\},g}$

$u_b, v_b, e_{\{u,v\},b}$

6) Given solution $S_{one\text{-}per}$ for one-per

⇒ solution $S_{3\text{-col}}$ = SolutionMap($S_{one\text{-}per}$) for 3-Col

Given $u_r = 1$, $u_g = 0$ ... solution

Colour u red if $u_r = 1$

blue if $u_b = 1$

green if $u_g = 1$

7) S_{super} valid $\Rightarrow$ S_{3col} valid

well defined colouring

$\forall$ node Because clause $u_r u_g u_b$ $\Rightarrow$ exactly one true

$\Rightarrow$ node u is given exactly one colour

$\forall$ edge + colour

clause $u_r v_r e_{uv,r}$ $\Rightarrow$ exactly one true

$\Rightarrow$ u_r + v_r not both true

$\Rightarrow$ u + v not both red

$\Rightarrow$ valid 3colouring

8) Reverse map

Given solution S_{3col}

$\Rightarrow$ give S_{super} = Reverse Solution Map(S_{3col})

$\forall$ nodes Given u is red ~~or~~ is green ...

Set $u_r = T$ $u_g = u_b = F$

$\forall$ edges $\{u, v\}$ + colours c

$e_{uv,c} = T$ iff $u_c \neq T$ + $v_c \neq T$

9) S_{3col} valid $\Rightarrow$ S_{super} valid

$\forall$ clause $u_r u_g u_b$ has exactly one True

because u has exactly one colour

$\forall$ clause $u_r v_r e_{uv,r}$ has exactly one True

because u + v cannot both be red

$u_r \neq T$ + $v_r \neq T$ $\Rightarrow$ $e_{uv,r} = T$

Making change NP-complete

1) What to reduce to

coins
Exact sum with repeats   Subset Sum = Exact sum without repeats

$\leq$ one per clause $\leq$ 3-SAT

2) Define Probs

One-per : Instance   n variables $x_1 \ldots x_n$
m clauses of 3 variables
no negated variables

Solution : A subset of variables to be True with exactly one variable true per clause

Making change

Instance : Coin denominations $c_1 < c_2 < \ldots < c_n$
Amount = A int
\# coins = k int

Solutions A set of at most k coins (repeats allowed) summing to exactly to A

a) Making change $\in$ NP

3) Reduction   One-per $\leq_p$ making change
Given alg for making change
Give alg for one-per

4) Similarities

oneper : Solution subset of variables to be true
making change : Solution subset of coins (need to use each at most once)

Give Alg for onepet

5) Given instance I_{onepet} for OnePET
⇒ instance I_{change} for makechange

I_{onepet}: n variables, m clauses

I_{change}: $K = n$

Do not mention has solution well defined

$A = 1\underbrace{000\ldots0}_{2\log n}\ 1\underbrace{00\ldots0}_{2\log n}\ 1 \ldots\ 1\underbrace{000\ldots0}_{2\log n}$

↑ clause 1 ↑ clause 2 clause m

∀ variable x_i there is a coin value c_i

$C_i = 000 \ldots 1\ 00 \ldots 1$

↖ put 1 in location corresponding to clause$_j$ when x_i is in clause$_j$

$C_{n+1} = 1$

6) Given solution S_{change} for make change
⇒ give $S_{3\text{-}SAT}$ for 3-SAT

Do not mention if valid well defined

S_{change} = set of coins

⇒ $S_{3\text{-}SAT}$ = set x_i = T if c_i is included (at least once)
= F else

7) Schnorr valid $\Rightarrow$ Sensor valid

Schnorr valid $\Rightarrow$ at most $k = n$ coins adding to A

$\xRightarrow{\text{lemma}}$ No carry possible in sum

$A = \underbrace{000000}_{2\log n} 1 \; 000 \ldots$

Coins $\quad 00000000 \; 1 \; 00000 \ldots$

15 such coins gives

$\underbrace{000000}\, 15 \; 00000$
Not correct sum

Need $> 10^{2\log n} >> n$ such coins to carry & give

$1 \underbrace{000000000}_{2\log n} 1$

But only $k = n$ coins allowed

$\therefore$ No carry

$\therefore \forall$ clause location $\quad 00010000$
$\quad\quad 1$
exactly one coin with 1
is included

$\Rightarrow$ exactly one variable in clause is true

8) Reverse map

Given solution $S_{coin\text{-}puzzle}$

$\Rightarrow$ Give solution S_{change}

If $\sigma = T$ $\Rightarrow$ include coin once

9) $S_{coin\text{-}puzzle}$ valid $\Rightarrow$ S_{change} valid

$\forall$ clause exactly one variable is T / true

$\Rightarrow$ $\forall$ clause place sum is 1

$= \sum c = A$

$\therefore$ coins $=$ # variables $= n$

3-Col $\leq$ Ind. Set

Similarities Instance G — Instance $\langle G, k \rangle$

4) Solution Subset of $\langle u,r \rangle$ $\longleftrightarrow$ Solution subset of nodes

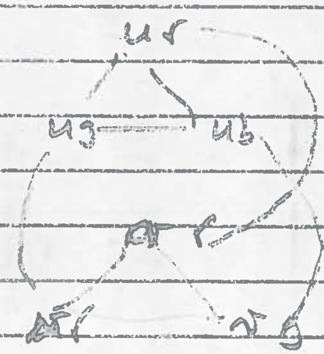
nodes not two edges
edges bichromatic

nodes taken
have no edges
between

5) Given Graph G to colour
Construct G' for Ind set

$\forall\, u \in G$ + colour r $\Rightarrow$ node $\langle u,r \rangle$

$\forall\, u$ edges



$\forall$ edge $\{u,v\}$

K = # nodes in G