York University
CSE 4111 Fall 2009
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Steps Uncomputable

1. Our goal is to prove that there are more real numbers than integers, i.e. $|\mathcal{R}| > |\mathcal{N}|$.
   We prove this by proving the following first order logic statement
   $\forall$ an inverse functions F^{-1} from $\mathcal{N}$ ideally to $\mathcal{R}$,
   $\exists x_{diagonal} \in \mathcal{R}, \forall i \in \mathcal{N}, F^{-1}(i) \neq x_{diagonal}$
   namely there are not enough integers to hit each real.
   We prove this by playing the game.
   Let F^{-1} be an arbitrary inverse function from $\mathcal{N}$ ideally to $\mathcal{R}$.
   Define the real $x_{diagonal} \in \mathcal{R}$ as follows.
   For each $i \in \mathcal{N}$, I must define the i^{th} digit of $x_{diagonal}$.
   For this, we use flip of the i^{th} diagonal element as follows.
   Let x_i denote the real $F^{-1}(i)$ that the i^{th} row gives us.
   Let d_i denote the i^{th} digit of x_i.
   Then let the i^{th} digit of $x_{diagonal}$ be any digit d'_i other than d_i.
   This completely defines $x_{diagonal}$.
   Continuing the game, let $i \in \mathcal{N}$ be arbitrary.
   Note $x_i = F^{-1}(i)$ and $x_{diagonal}$ differ in their i^{th} digits.
   This proves that $F^{-1}(i) \neq x_{diagonal}$.

2. Our goal is to prove that there is an uncomputable computation problem P_{hard},
   i.e. one for which each TM M fails to compute,
   because there in an input I_M on which it gives the wrong answer, i.e. $M(I_M) \neq P_{hard}(I_M)$.
   This is stated using the first order logic statement:
   $\exists P_{hard}\ \forall M\ \exists I_M\ M(I_M) \neq P_{hard}(I_M)$
   We prove this using the game.
   Define P_{hard} to be the problem $\neg Problem_{diagonal}$, defined as
   $\neg Problem_{diagonal}(\text{“}M\text{”}) = 0$ iff $M(\text{“}M\text{”}) = 1$, i.e. M on “M” halts and says “yes”
   (assuming “M” is a valid the description of TM M).
   Continuing the game, let M be an arbitrary TM.
   Define input I_M to be the description “M” of TM M.
   We know M does not accept $\neg Problem_{diagonal}$,
   because it gives the wrong answer on input $I_M =$ “M”,
   i.e. $M(I_M) \neq P_{hard}(I_M)$.
   This completes the proof that there is an uncomputable computation problem.