

CSE 3101 Design and Analysis of Algorithms

Solutions for Practice Test for Unit 1

Loop Invariants and Iterative Algorithms

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1. **Tiling Chess Board:** You are given a $2n$ by $2n$ chess board. You have many tiles each of which can cover two adjacent squares. Your goal is to place non-overlapping tiles on the board to cover each of the $2n \times 2n$ tiles except for the top-left corner and the bottom-right corner. Prove that this is impossible. To do this give a loop invariant that is general enough to work for any algorithm that places tiles. Hint: chess boards color the squares black and white.

- Answer:

Precond: The board is empty.

Postcond: All but the corner two white tiles is covered.

Define Loop Invariant: The number of blacks and white tiles covered is equal.

Establishing the Loop Invariant: $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$: By $\langle pre-cond \rangle$ there are zero white and zero blacks covered. Hence, these numbers are the same.

Step: Place a tile. This covers one white and one black square.

Maintaining the Loop Invariant: $\langle loop-invariant' \rangle \ \& \ not \ \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$: By $\langle loop-invariant' \rangle$, at the beginning of the iteration the number of white and black tiles covered are equal. The step covers one more white and one more black. Hence, after the iteration the numbers are still equal.

Exit: Exit when ever you like.

Ending: Obtaining that the postcondition is NEVER met.

i.e. $\langle loop-invariant \rangle \Rightarrow \neg \langle post-cond \rangle$

Prove the contra positive,

i.e. $\langle post-cond \rangle \Rightarrow \neg \langle loop-invariant \rangle$:

Assume that the post condition is true. It states that all but the corner two white tiles is covered. However, the board has an equal number of black and white tiles. Hence, the number of white and black tiles covered are not equal. Hence, the LI is not true.

2. **Longest Contiguous Increasing Subsequence (LCIS):** The input consists of a sequence $A[1..n]$ of integers and we want to find a longest contiguous subsequence $A[k_1..k_2]$ such that the elements are *strictly* increasing. For example, the optimal solution for $[5, 3, 1, 3, 7, 7, 9, 8]$ is $[1, 3, 7]$.

- (a) Provide a tight lower bound on the running time for this problem. Prove that your answer is indeed a tight lower bound.

- Answer: The problem takes at least $\Omega(n)$ time because the input needs to be read. Let A be an arbitrary algorithm. Give it the input consisting of all zero. If it does not read some integer, change this integer to a 1 (If it is the first integer change it to -1). It answers the same longest length in both cases, however, in the first the longest has length one and in the second two. Therefore, it is wrong in at least one of these cases.

- (b) Specify an appropriate loop-invariant, measure of progress and exit condition for an iterative solution to this problem. No explanations are necessary.

- Answer: LI: We maintain an over all optimal LCIS found so far and the LSIS that we are currently working on.

$A[k_1 \dots k_2]$ is an LCIS of $A[1 \dots i]$ and $A[p \dots i]$ is the LCIS of $A[1 \dots i]$ that ends at i .

Note the first LI is "More of the Input", namely if we consider the prefix $A[1 \dots i]$ to be the entire input then we have the solution.

Most people missed the second one, but it is needed if the LI is to be maintained.

Measure of Progress: the number i of elements read. Many gave that i increases. This is progress, but not a measure.

Exit Condition: When all the elements have been read because $i = n$.

- (c) Provide concise pseudo-code for an algorithm $\text{LCIS}(A,n)$ that returns the indices k_1, k_2 of the LCIS of $A[1 \dots n]$ uses the loop-invariant, measure of progress and exit condition you specified above. Assume $n \geq 1$.

- Answer:

I thought more about why LI are important to me. It is a life philosophy. It is about feeling grounded. Most of the code I marked today makes me feel ungrounded. It cycles, but I don't know what the variables mean, how they fit together, where the algorithm is going, or how to start thinking about it. Loop invariants are about starting my day at home, where I know what is true and what things mean. Loop invariants are also about returning back full circle back to my safe home at the end of my day

algorithm $[k_1, k_2] = \text{LCIS}(A,n)$

⟨pre-cond⟩: $A[1 \dots n]$ is an array of integers

⟨post-cond⟩: $A[k_1 \dots k_2]$ is an LCIS of $A[1 \dots n]$

begin

$k_1 = k_2 = p = i = 1$

loop *⟨loop-invariant⟩*: $A[k_1 \dots k_2]$ is an LCIS of $A[1 \dots i]$ and
 $A[p \dots i]$ is the LCIS of $A[1 \dots i]$ that ends at i

$i = i + 1$

if($A[i-1] \geq A[i]$) then

$p = i$

end if

if $(q - p) > (k_2 - k_1)$

$k_1 = p$

$k_2 = q$

end if

end if

end algorithm

- (d) Provide an informal argument for how your code is able to maintain the loop invariant. A formal proof is not required.

- Answer: $\langle \text{loop-invariant}' \rangle \ \& \ \text{not} \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{loop}} \Rightarrow \langle \text{loop-invariant}'' \rangle$

Suppose at the beginning of an iteration, the LI is true for the current values, i.e. $A[k'_1 \dots k'_2]$ is an LCIS of $A[1 \dots i']$ and $A[p' \dots i']$ is the LCIS of $A[1 \dots i']$ that ends at i' .

The first block of code either extends the LCIS that we are currently working on from $A[p' \dots i']$ to $A[p' \dots i'+1]$ or shrinks it to contain only the one element $A[i'+1]$ depending on whether or not $A[i'+1]$ is bigger than $A[i']$. This is achieved by moving its end point from i' to $i'' = i'+1$ and either leaving its begging at $p'' = p'$ or moving it to $p'' = i''$. Either way, the LI is maintained that $A[p'' \dots i'']$ is the LCIS of $A[1 \dots i'']$ that ends at i'' .

The second block of code updates the over all LCIS found so far simply by moving it to the current one, if the current one has grown to be larger. Either way, the LI is maintained that $A[k''_1 \dots k''_2]$ is an LCIS of $A[1 \dots i'']$

- (e) What are the other key steps in proving your algorithm correct? List and provide a concise proof for each.

- Answer:

Establishing the LI: $\langle \text{pre-cond} \rangle \ \& \ \text{code}_{\text{pre-loop}} \Rightarrow \langle \text{loop-invariant} \rangle$.

On entry, $k_1 = k_2 = p = i = 1$. Thus the first part of the loop-invariant requires that $A[1]$ be an LCIS of $A[1]$, which is trivially true. The second part of the loop invariant requires that $A[1]$ is the LCIS of $A[1]$ that ends at location 1, which is also trivially true.

Making Progress: i increments by 1 on each iteration. Program exits when $i = n$.

Establishing Post-Condition: $\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$. When $i = n$, the first LI gives that $A[k_1 \dots k_2]$ is an LCIS of $A[1 \dots n]$, which is the post condition.

3. **$d + 1$ Colouring:** Given an undirected graph G such that each node has at most d neighbors, colour each node with one of $d + 1$ colours so that for each edge the two nodes have different colours. Hint: Don't think too hard. Just colour the nodes. What loop invariant do you need?
Hint: All it will say is that you have not gone wrong yet.
Give all the steps done in class to develop this iterative algorithm.

- Answer:

Loop Invariant The only loop invariant that you need is that after colouring i nodes, you have not yet coloured the two nodes of an edge with the same colour.

Establishing the Loop Invariant $\langle \text{pre-cond} \rangle \ \& \ \text{code}_{\text{pre-loop}} \Rightarrow \langle \text{loop-invariant} \rangle$:
 $\text{code}_{\text{pre-loop}}$: Initially, $i = 0$ nodes have been coloured, so the LI is trivially true.

Steps: Grab any uncoloured node v . It has at most d neighbors some of which might already be coloured. However, at most d different colours different colours can be already used by these neighbors. Colour v with one of the other $d + 1$ colours.

Maintaining the Loop Invariant $\langle \text{loop-invariant}' \rangle \ \& \ \text{not} \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{loop}} \Rightarrow \langle \text{loop-invariant}'' \rangle$:

By the LI', we had coloured i nodes without any bichromatic edges. We proved that the step manages to colour and $i + 1^{\text{st}}$ node so that it has a different colour than its neighbors. This reestablishes the LI with $i + 1$ nodes coloured.

Measure of Progress and Time: The measure of progress is the number of coloured nodes. Each of the n iterations will take at most $O(d)$ time for a total of at most $O(dn)$ time.

Obtaining the postcondition $\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$:

By exit condition, $i = n$.

By LI, all these nodes have been coloured without bichromatic edges.

Hence we have coloured the entire graph without them. This establishes the post condition.

In Sides or Videos

4. (Answer in slides)

Iterative Cake Cutting: The famous algorithm for fairly cutting a cake in two is for one person to cut the cake in the place that he believes is half and for the other person to choose which "half" he likes. One player may value the icing more while the other the cake more, but it does not matter. The second player is guaranteed to get a piece that he considers to be worth at least a half because he choose between two pieces whose sum worth for him is at least a one. Because the first person cut it in half according to his own criteria, he is happy which ever piece is left for him. Our goal is write an iterative algorithm which solves this same problem for n players.

To make our life easier, we view a cake not as three dimensional thing, but as the line from zero to one. Different players value different subintervals of the cake differently. To express this, he assigns some numeric value to each subinterval. For example, if player p_i 's name is written on the subinterval $[\frac{i-1}{2n}, \frac{i}{2n}]$ of cake then he might allocate a higher numeric value to it, say $\frac{1}{2}$. The only requirement is that the sum total value of the cake is one.

Your algorithm is only allowed the following two operations. In an evaluation query, $v = \text{Eval}(p, [a, b])$, the algorithm asks a player p how much v he values a particular subinterval $[a, b]$ of the whole cake

$[0, 1]$. In a cut query, $b = \text{Cut}(p, a, v)$, the protocol asks the player p to identify the shortest subinterval $[a, b]$ starting at a given left endpoint a , with a given value v . In the above example, $\text{Eval}(p_i, [\frac{i-1}{2n}, \frac{i}{2n}])$ returns $\frac{1}{2}$ and $\text{Cut}(p_i, \frac{i-1}{2n}, \frac{1}{2})$ returns $\frac{i}{2n}$. Using these the two player algorithm is as follows.

algorithm *Partition2*($\{p_1, p_2\}, [a, b]$)

<pre-cond> p_1 and p_2 are players.

$[a, b] \subseteq [0, 1]$ is a subinterval of the whole cake.

<post-cond> Returns a partitioning of $[a, b]$ into two disjoint pieces $[a_1, b_1]$ and $[a_2, b_2]$ so that player p_i values $[a_i, b_i]$ at least half as much as he values $[a, b]$.

begin

$v_1 = \text{Eval}(p_1, [a, b])$

$c = \text{Cut}(p_1, a, \frac{v_1}{2})$

if($\text{Eval}(p_2, [a, c]) \leq \text{Eval}(p_2, [c, b])$) then

$[a_1, b_1] = [a, c]$ and $[a_2, b_2] = [c, b]$

else

$[a_1, b_1] = [c, b]$ and $[a_2, b_2] = [a, c]$

end if

return($[a_1, b_1]$ and $[a_2, b_2]$)

end algorithm

The problem that you must solve is the following

algorithm *Partition*(n, P)

<pre-cond> P is a set of n players.

Each player in P values the whole cake $[0, 1]$ by at least one.

<post-cond> Returns a partitioning of $[0, 1]$ into n disjoint pieces $[a_i, b_i]$ so that for each $i \in P$, the player p_i values $[a_i, b_i]$ by at least $\frac{1}{n}$.

begin

...

end algorithm

- (a) Can you cut off n pieces of cake, each of *size* strictly bigger than $\frac{1}{n}$, and have cake left over? Is it sometimes possible to allocated a disjoint piece to each player, each worth by the receiving player much more than $\frac{1}{n}$, and for there to still be cake left? Explain.
- (b) Give **all** the required steps to describe this LI algorithm. (Even if you do not know how to do a step for this algorithm, minimally state the step.)

As a big hint to designing an iterative algorithm, we will tell you what the first iteration accomplishes. (Later iterations may do slightly modified things.) Each player specifies where he would cut if he were to cut off the first $\frac{1}{n}$ fraction of the $[a, b]$ cake. The player who wants the smallest amount of this first part of the cake is given this piece of the cake. The code for this is as follows.

loop $i \in P$

$c_i = \text{Cut}(p_i, 0, \frac{1}{n})$

end loop

i_{min} = the $i \in P$ that minimizes c_i

$[a_{i_{min}}, b_{i_{min}}] = [0, c_{i_{min}}]$

As a second big hint, your loop invariant should include:

- i. How the cake has been cut so far.

- ii. Who has been given cake and how do they feel about it.
- iii. How do the remaining players feel about the remaining cake.

• Answer:

- (a) Extra question: Clearly the first task is impossible because summing up n things of size more than $\frac{1}{n}$ gives you a total more than one. There can be none left over. However, the second is possible. Consider the example given above in which for each i , $Eval(p_i, [\frac{i-1}{2n}, \frac{i}{2n}])$ returns $\frac{1}{2}$. This subinterval $[\frac{i-1}{2n}, \frac{i}{2n}]$ can be allocated to this player. These intervals are disjoint, valued much more than $\frac{1}{n}$ by this player, and the interval $[\frac{1}{2}, 1]$ is left over.

(b) Define Loop Invariant:

- i. First remember the *more of the output* type of loop invariant. This would be as follows. There is a set of players Q that have been served. The beginning $[0, a]$ of the cake has been partitioned into $|Q|$ disjoint pieces. Each player $p_i \in Q$ has been allocated a piece $[a_i, b_i]$ worth at least $\frac{1}{n}$ to him.
- ii. Now remember the *Fairy God Mother* type of loop invariant. I have arrived from Mars, some amount of progress has been made already and my Fairy God Mother has set the state of the computation to be just what I want so that I feel as comfortable as I could be given the amount of progress that has been made. This would be as follows. The remaining $[a, 1]$ interval of the cake is worth at least $\frac{n-|Q|}{n}$ to each of the remaining players, i.e. to those in $P - Q$.

This is a more of the output loop invariant if you think of the output being ordered by who gets the first piece, who gets the second, and so on.

- (c) Establishing the Loop Invariant $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$:
With $a = 0$ and $Q = \emptyset$, the beginning zero $[0, 0]$ of the cake has been partitioned into disjoint pieces and allocated to $|Q| = 0$ of the players. All $[0, 1]$ of the cake remains and by the precondition is worth at least $\frac{n-|Q|}{n} = 1$ to each of the players.

(d) Define the Step (actually I will give the entire code):

algorithm *Partition*(P)

$\langle pre-cond \rangle$: As above

$\langle post-cond \rangle$: As above

begin

$a = 0$ and $Q = \emptyset$

loop **$\langle loop-invariant \rangle$** : As above.

exit when $|Q| = n$

loop $i \in P - Q$

$c_i = Cut(p_i, a, \frac{1}{n})$

end loop

$i_{min} =$ the $i \in P - Q$ that minimizes c_i

$[a_{i_{min}}, b_{i_{min}}] = [a, c_{i_{min}}]$

$a = c_{i_{min}}$

$Q = Q + i_{min}$

end loop

return all parts $[a_i, b_i]$ for each $i \in P$

end algorithm

- (e) Maintaining the Loop Invariant $\langle loop-invariant' \rangle \ \& \ not \ \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$:

Let a' and P' be the values at the top of the loop and let a'' and Q'' be the values after going around again. By the code, $a'' = c_{i_{min}}$ and $Q'' = Q' + i_{min}$.

- i. By $\langle loop-invariant' \rangle$ the beginning $[0, a']$ of the cake has been partitioned into $|Q'|$ disjoint pieces. By the code, we just partitioned off the piece $[a', a'']$. Hence, the beginning $[0, a'']$ of the cake has been partitioned into $|Q'| + 1 = |Q''|$ disjoint pieces.

- ii. By $\langle \text{loop-invariant}' \rangle$, each player $p_i \in Q'$ has been allocated a piece worth at least $\frac{1}{n}$ to him. We just allocated a piece of this value to another player and added him to Q' . Hence, each player $p_i \in Q''$ has been such a piece.
- iii. Consider a remaining player p_i in $P - Q'' = P - Q' - i_{min}$. Because he marked it, we know that this player values the piece $[a', c_i]$ by $\frac{1}{n}$. The piece $[a', c_{i_{min}}] = [a', a'']$ cut off is smaller than this, because $c_{i_{min}}$ is smaller than c_i . Hence, player p_i values the piece $[a, a'']$ by at most $\frac{1}{n}$. By $\langle \text{loop-invariant}' \rangle$, the remaining $[a', 1]$ interval of the cake is worth at least $\frac{n - |Q'|}{n}$ to p_i . Hence, $[a'', 1] = [a', 1] - [a', a'']$ has value of at least $\frac{n - |Q'|}{n} - \frac{1}{n} = \frac{n - |Q''|}{n}$.

It follows that $\langle \text{loop-invariant}'' \rangle$ is true and that the loop invariant has been maintained.

- (f) Define Exit Condition: When every player has a piece we exit.
- (g) Obtaining the postcondition $\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$:
By the exit condition $|Q| = n$. By the loop invariant, the beginning $[0, a]$ of the cake has been partitioned into $|Q| = n$ disjoint pieces and each player $p_i \in Q = P$ has been allocated a piece $[a_i, b_i]$ worth at least $\frac{1}{n}$ to him. Though there may be some cake left over, the post condition is satisfied.
- (h) Measure of Progress: The measure of progress is number of pieces allocated. Each iteration one piece is allocated. Initially, there are only n pieces to be allocated. Hence, with sufficient progress, the exit condition will be met. The i^{th} of the n iterations takes $\mathcal{O}(n-i)$ time for a total running time of $\mathcal{O}(n^2)$.

5. (Answer in slides)

Sorted Matrix Search: Search a Sorted Matrix Problem: The input consists of a real number x and a matrix $A[1..n, 1..m]$ of nm real numbers such that each row $A[i, 1..m]$ is sorted from left to right and each column $A[1..n, j]$ is sorted from top to bottom. The goal is to determine if the key x appears in the matrix. Design and analyze an iterative algorithm for this problem that examines as few matrix entries as possible. Careful, if you believe that a simple binary search solves the problem. Later we will ask for a lower bound and for a recursive algorithm.

- Answer: Doing binary search in $\mathcal{O}(\log(n \times m))$ time is impossible. See the lower bound question. If you take $\mathcal{O}(n \log m)$ time doing binary search in each row than you are taking too much time. It can be done by examining $n + m - 1$ entries. Observe that the values in the matrix increase from $A[1, 1]$ to $A[n, m]$. Hence, the boundary between values that are less than or equal to x and those that are greater follows some monotonic path from $A[1, m]$ to $A[n, 1]$. The algorithm traces this path starting at $A[1, m]$. When it is at the point $A[i, j]$, the loop invariant is that we have stored the best answer from those outside of the sub-rectangle $A[i..n, 1..j]$. Initially, this is true for $[i, j] = [1, m]$, because none of the matrix is excluded. Now suppose it is true for an arbitrary $[i, j]$. The algorithm then compares $A[i, j]$ with x . If it is better than our current best answer then our current best is replaced. If $A[i, j] \leq x$, then because the values in the row $A[i, 1..j]$ are all smaller or equal to $A[i, j]$, these are worse answers and hence we can conclude that we now have the best answer from those outside of the sub-rectangle $A[i + 1..n, 1..j]$. We maintain the loop invariant by increasing i by one. On the other hand, if $A[i, j] > x$, then it is too big and so are all the elements in the column $A[i..n, j]$ which are even bigger. We can conclude that we have the best answer from those outside of the sub-rectangle $A[i..n, 1..j - 1]$. We maintain the loop invariant by decreasing j by one. The exit condition is $|i..n| = 0$ or $|1..j| = 0$ (i.e. $i > n$ or $j < 1$). When this occurs, the sub-rectangle $A[i..n, 1..j]$ is empty. Hence, our best answer, which by the loop invariant is the best from those outside of the this sub-rectangle, must be the best overall. The measure of progress $|i..n| + |1..j| - 1 = (n - i + 1) + (j) - 1$ is initially $n + m - 1$ and decrease by one each iteration. After $n + m - 1$ iterations, either the algorithm has already halted or the measure has reached zero at which point the exit condition is definitely met.

6. (Answer in slides)

Connected Components: The input is a matrix I of pixels. Each pixel is either black or white.

A pixel is considered to be connected to the four pixels left, right, up, and down from it, but not diagonal to it. The algorithm must allocate a unique name to each connected black component. (The name could simply be 1,2,3,..., to the number of components.) The output consists of another matrix $Names$ where $Names(x, y) = 0$ if the pixel $I(x, y)$ is white and $Names(x, y) = i$ if the pixel $I(x, y)$ is a part of the component named i . The algorithm reads the matrix from a tape row by row as follows.

```

loop y = 1 . . . h
  loop x = 1 . . . w
    ⟨loop-invariant⟩: ??
    if(  $I(x, y) = \text{white}$  )
       $Names(x, y) = 0$ 
    else
      ???
    end if
  end loop
end loop
end algorithm

```

The image may contain spirals and funny shapes. Connected components may contain holes that contain other connected components. A particularly interesting case is the image of a comb consisting of many teeth held together at the bottom by a handle. Scanning the image row by row, one would first see each tooth as a separate component. As the handle is read, these teeth would merge into one.

(a) Give the classic *more of the input loop invariant* algorithm. Don't worry about its running time.

- Answer: Assuming that the input image consists only of the part of the image read in so far, the algorithm has named the connected components. This loop invariant is maintained while making progress as follows. The next pixel $I(x, y)$ is read. The white case is easy, so suppose it is black. This pixel is connected to two pixels that have been read before: $I(x-1, y)$ to its left and $I(x, y-1)$ above it. If these are both white, then $I(x, y)$ is in a new component by itself and hence is given the new name never used before. If only one of $I(x-1, y)$ and $I(x, y-1)$ is black or the two are black but in the same component, then $I(x, y)$ becomes a part of this component. Finally, if $I(x-1, y)$ and $I(x, y-1)$ are both black and in the two different components, then the new pixel $I(x, y)$ joins these two components into one. Suppose $I(x-1, y)$ has been named i and $I(x, y-1)$ named i' . Every pixel in the component named i' is renamed to i .

```

maxName = numberOfComponents = 0

```

```

  loop y = 1 . . . h

```

```

    loop  $x = 1 \dots w$  ⟨loop-invariant⟩: More of the input.

```

```

      if(  $I(x, y) = \text{white}$  )
         $Names(x, y) = 0$ 

```

```

      else

```

```

        if(  $Names(x, y-1) = 0$  and  $Names(x-1, y) = 0$  )

```

```

          ++ maxName

```

```

          ++ numberOfComponents

```

```

           $Names(x, y) = \text{maxName}$ 

```

```

        elseif(  $Names(x, y-1) = 0$  or  $Names(x-1, y) = 0$  or  $(Names(x, y-1) = Names(x-1, y))$  )

```

```

           $Names(x, y) = \text{the nonzero one of } Names(x, y-1) \text{ and } Names(x-1, y)$ 

```

```

        else

```

```

           $i = Names(x, y-1)$ 

```

```

           $i' = Names(x-1, y)$ 

```

```

           $Names(x, y) = i$ 

```

```

        Replace every  $i'$  in  $Names$  with  $i$ .
        --  $numberOfComponents$ 
    end if
end if
end loop
end loop
end algorithm

```

- (b) This version of the question is easier in that the matrix $Names$ need not be produced. The output is simply the number of connected black components in the image. However, this version of the question is harder in your computation is limited in the amount of memory it can use. For example, you don't remember pixels that you have read if you do not store them in this limited memory and you don't have nearly enough memory to store them all. The number of components may be $\Theta(\text{the pixels})$ so you cant store all of them either. How little memory can you get away with?

- Answer: The amount of memory needed is $\Theta(w)$. The algorithm will be the same except it only keeps the last two rows worth of $Names$. Note $numberOfComponents$ has already been added to the code to count the number of components.

- (c) In this this final version, in addition to a small amount of fast memory, you have a small number of tapes for storing data. Data access on tapes, however, is quite limited. A tape is in one of three modes and cannot switch modes mid operation. In the first mode, the data on a tape is read forwards one data item at a time in order. The second mode, is the same except it is read backwards. In the third mode, the tape is written to. However, the command is simply $write(data)$ which appends this data to the end of the tape. Data on a tape cannot be changed. All you can do is to erase the entire tape and start writing again from the beginning. An algorithm must consist of a number of passes. The first pass reads the input from the input tape one pixel at a time row by row in order. As it goes, the algorithm updates what is store in fast memory and outputs data in order onto a small number output tapes. Successive passes can read what was written during a previous pass and/or the input again. The last pass must write the required output $Names$ onto a tape. You want to use as little fast memory and as few passes as possible. For each pass clearly state the loop invariant.

- Answer: The algorithm will have two passes, two intermediate tapes, and $\Theta(w)$ fast memory. During the first pass, the algorithm will again be the same. As in the last algorithm, it only keeps the last two rows worth of $Names$ in the fast memory. As rows of $Names$ is deleted from fast memory it is written to a tape $Names_{firsttry}$. The problem is that as components named i' are renamed to i , this updating will not occur on the tape. Hence, $Names_{firsttry}$ may have many names associated with each component. To keep track of these repeated names, every time the algorithm would like to rename i' to i , the instruction $\langle i' \leftarrow i \rangle$ is written to a second tape $Instructions$.

The loop invariant for this first pass states that what is in the fast memory is exactly the same as it would be in the second algorithm's memory and if all the instructions $\langle i' \leftarrow i \rangle$ in $Instructions$ are followed, then what is currently in the tape $Names_{firsttry}$ would be converted into what would be in the first algorithm's $Names$. More over, the instruction $\langle i' \leftarrow i \rangle$ appears in the location of the tape $Instructions$ corresponding to the last place in $Names_{firsttry}$ that the name i' appears. Hence, if one reads both tapes $Names_{firsttry}$ and $Instructions$ backwards, then one would always have the required set of instructions to do the required conversion.

The second pass reads these two tapes backwards. Its loop invariant is that it has already correctly converted the part of $Names_{firsttry}$ that is has seen and stored it backwards in the final output tape $Names$. The loop invariant also states that fast memory contains the needed instructions to convert the next pixel. The step is to read the next pixel from $Names_{firsttry}$, read any new instructions from $Instructions$, convert this next pixel using the current set of instructions, and write the converted pixel into $Names$.

One concern might be that the number of instructions read may exceed the size of the fast memory. However, as soon as a name i' no longer appears in a row of $Names_{firsttry}$, we know that it will never appear again earlier in $Names_{firsttry}$. Hence, at this point the instruction $\langle i' \leftarrow i \rangle$ can be deleted. This ensures, that the number of instructions in fast memory never exceeds the number w of pixels in a row.

7. (Answer in videos)

A *tournament* is a directed graph (see Section ??) formed by taking the complete undirected graph and assigning arbitrary directions on the edges, i.e., a graph $G = (V, E)$ such that for each $u, v \in V$, exactly one of $\langle u, v \rangle$ or $\langle v, u \rangle$ is in E . A *Hamiltonian path* is a path through a graph that can start and finish any where but must visit every node exactly once each. Design an algorithm which finds a Hamiltonian path through it given any tournament. Because this algorithm finds a Hamiltonian path for each tournament, this algorithm, in itself, acts as proof that every tournament has a Hamiltonian path.

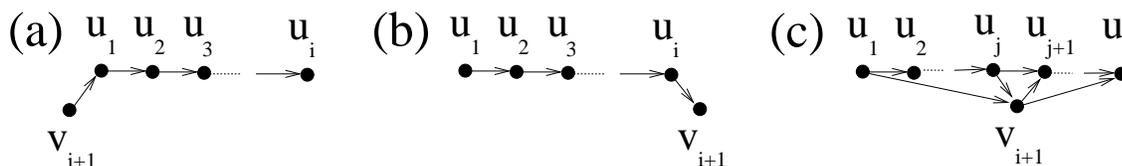
- Answer:

More of the Output: It is natural to want to push forward and find the required path through a graph. The measure of progress would be the amount of the path output and the loop invariant would say “I have the first i nodes (edges) in the final path.” Maintaining this loop invariant would require extending the path constructed so far by one more node. The problem, however, is that the algorithm might get stuck, when the path constructed so far has no edges leaving the last node to a node that has not yet been visited. This makes the loop invariant as stated false.

Recursive Backtracking: One is then tempted to have the algorithm “backtrack” when it gets stuck trying in a different direction for the path to go. This is a fine algorithm. See recursive backtracking algorithms in Chapter ?. However, unless one is really careful, such algorithms tend to require exponential time.

More of the Input: Instead, try solving this problem using a “more of the input” loop invariant. Assume the nodes are numbered 1 to n in an arbitrary way. The algorithm temporarily pretends that the sub-graph on the first i of the nodes is the entire input instance. The loop invariant is “I currently have a solution for this sub-instance.” Such a solution is a Hamiltonian path u_1, \dots, u_i that visits each of the first i nodes exactly once each, which in turn is simply a permutation the first i nodes. Maintaining this loop invariant requires constructing a path for the first $i + 1$ nodes. There is no requirement that this new path resembles the previous path. However, for this problem, it can be accomplished by finding a place to insert the $i + 1^{st}$ node within the permutation of the first i nodes. In this way, the algorithm looks a lot like insertion sort.

Case Analysis: When developing an algorithm, a good technique is to see for which input instances the obvious thing works and then try to design another algorithm for the remaining cases.



- (a) If $\langle v_{i+1}, u_1 \rangle$ is an edge, then the extended path is easily v_{i+1}, u_1, \dots, u_i .
- (b) Similarly, if $\langle u_i, v_{i+1} \rangle$ is an edge, then the extended path is easily u_1, \dots, u_i, v_{i+1} .
- (c) Otherwise, because the graph is a tournament, both $\langle u_1, v_{i+1} \rangle$ and $\langle v_{i+1}, u_i \rangle$ are edges. Color each node u_j red if $\langle u_j, v_{i+1} \rangle$ is an edge and blue if $\langle v_{i+1}, u_j \rangle$ is. Because u_1 is red and u_i is blue, there must be some place u_j to u_{j+1} in the path where path changes color from red to blue. Because both $\langle u_j, v_{i+1} \rangle$ and $\langle v_{i+1}, u_{j+1} \rangle$ are edges, we can form the extended path $u_1, \dots, u_j, v_{i+1}, u_{j+1}, \dots, u_i$.

8. (Answer in videos)

An *Euler tour* in an undirected graph is a cycle that passes through each edge exactly once. A graph contains an Eulerian cycle iff it is connected and the degree of each vertex is even. Given such a graph find such a cycle.

- Answer:

More of the Output: We will again start by attempting to solve the problem using the more the output technique, namely, start at any node and build the output path one edge at a time. Not having any real insight into which edge should be taken next, we will choose them in a blind or “greedy” way (see Chapter ??). The loop invariant is that after i steps you have some path through i different edges from some node s to some node v .

Getting Stuck: The next step in designing this algorithm is to determine when, if ever, this simple blind algorithm gets stuck and to either figure out how to avoid this situation or to fix it.

Making Progress: If $s \neq v$, then the end node v must be adjacent to an odd number of edges that are in the path. See Figure ??a. This is because there is the last edge in the path and then for every edge in the path coming into the node there is one leaving. Hence, because v has even degree it follows that v is adjacent to at least one edge that is not in the path. Follow this edge extending the path by one edge. This maintains the loop invariant while making progress. This process can only get stuck when the path happens to cycle around back to the starting node giving $s = v$. In such a case, join the path here to form a cycle.

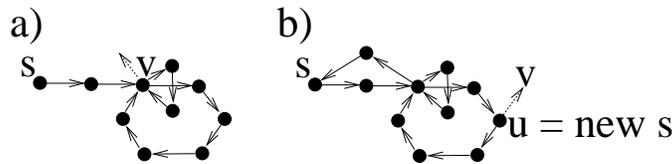


Figure 1: Euler Algorithm

Ending: If the cycle created covers all of the edges, then we are done.

Getting Unstuck: If the cycle we have created from our chosen node s back to s does not cover all the edges, then we look for a node u with in this cycle that is adjacent to an edge not in the cycle. See Figure ??b. Change s to be this new node u . We break the cycle at u giving us a path from u back to u . The difference with this path is that we can extend it past u along this unvisited edge. Again the loop invariant has been maintained while making progress.

u Exists: The only thing remaining to prove is that when v comes around to meet s again and we are not done, then there is in fact a node u in the path that is adjacent to an edge not in the path. Because we are not done then there is an edge e in the graph that is not in our path. Because the graph is connected, there must be a path in the graph from e to our constructed path. The node u at which this connecting path meets our constructed path must be as required because the last edge $\{u, w\}$ in the connecting path is not in our constructed path.

Extended Loop Invariant: To avoid having to find such a node u when it is needed, we extend the loop invariant to state that in addition to the path, the algorithm remembers some node u other than s and v that is in the path and is adjacent to an edge not in the path.

J: Except for this little implementation detail, this feels like more of the

In Lectures

9. Pointers (Answer in slides)

“And the last shall be first!”

(For each of A-K see the figure on the next page labeled with the corresponding letter. Here $n = 6$.)

A: Precondition; The input to this problem is a linked list of n nodes in which the last node points back to the first forming a circle. The variable *last* points at the “last” node in the list. The nodes are of type *Node* containing a field *info* and a field *next* of type *Node* which is to point at the next node in the list. The values 1, 2, . . . , 6 in figure are just to help you and cannot be used in the code.

B: Postcondition; The required output to this problem consists of the same nodes in the same circle, except each node now points at the “previous” node instead of the “next”. In other words the pointers are turned from clockwise to counter clockwise. The variable *last* still points at the same node, but this “last” node has become “first”.

C: Loop Invariant_t; Your algorithm for this problem will be iterative (i.e. a loop taking one step at a time). Your first task is to give the loop invariant. This is not done in words but by **drawing** what the data structure will look like after the algorithm has executed its loop $t = 2$ times. Hint: Nodes with value 1 and 2 have their pointers fixed. Hint: You may need a couple of additional pointers to hold things in place. Give them as meaningful names as possible.

D: Loop Invariant_{t+1}; The next task in developing an iterative algorithm is:

- Maintain the Loop Invariant:

We arrived at the top of the loop knowing only that the Loop Invariant is true and the Exit Condition is not.

We must take one step (iteration) (making some kind of progress).

And then prove that the Loop Invariant will be true when we arrive back at the top of the loop.

$\langle \text{loop-invariant}_t \rangle \ \& \ \text{not} \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{loop}} \Rightarrow \langle \text{loop-invariant}_{t+1} \rangle$

Towards this task, **draw** what the data structure will look like after the algorithm has executed its loop one more time, i.e. $t+1 = 3$ times.

E: Code C to D; In space E, give me the code to change the data structure in figure C into that in figure D. This will be the code within the loop that makes progress while maintaining the loop invariant. Assume every variable and field is public.

F: Loop Invariant₀; The next task in developing an iterative algorithm is:

- Establish the Loop Invariant:

Our computation has just begun and all we know is that we have an input instance that meets the Pre Condition.

Being lazy, we want to do the minimum amount of work.

And to prove that it follows that the Loop Invariant is then true.

$\langle \text{pre-cond} \rangle \ \& \ \text{code}_{\text{pre-loop}} \Rightarrow \langle \text{loop-invariant} \rangle$

Towards this task, **draw** what the data structure will look like when the algorithm is at the top of the loop but has executed this loop zero times. Make it as similar as possible to the precondition so that minimal work needs to be done in G.

G: Code A to F; In space G, give me the code to change the data structure in figure A into that in figure F. This will be the initial code that establishes the loop invariant.

H: Loop Invariant_n; The next task in developing an iterative algorithm is:

- Obtain the Post Condition:

We know the Loop Invariant is true because we have maintained it.

We know the Exit Condition is true because we exited.

We do a little extra work.

And then prove that it follows that the Post Condition is true.

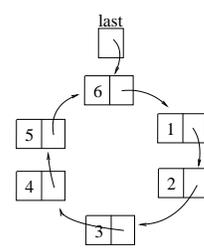
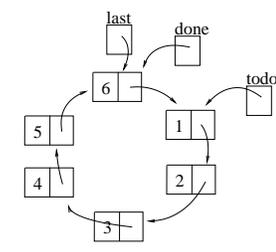
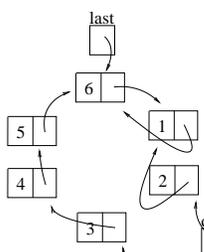
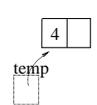
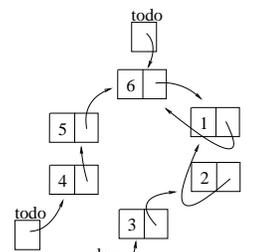
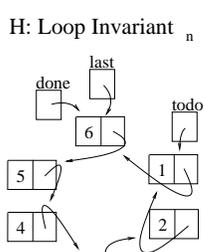
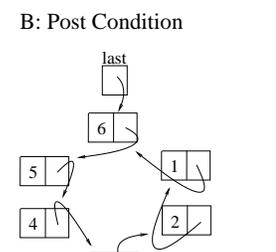
$\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$

Towards this task, **draw** what the data structure will look like after the algorithm has executed it's loop n times. Make it as similar as possible to the postcondition so that minimal work needs to be done in J.

I: Exit Condition: What is the exit condition, i.e. how does your code recognize that it is in the state you drew in H, given that the algorithm does not know n (and did not count t) and does not know the values 1, 2, ..., 6. Also be careful that your exit condition does not have you drop out at state D.

J: Code H to B; In space J, give me the code to change the data structure in figure H into that in figure B. This will be the final code that establishes the post condition.

K: Give the complete code; Put all the code together into a routine solving the problem.

| | | | |
|---|--|---|--|
| <p>A: Precondition</p>  | <p>G: Code A to F</p> <pre>Node done = last; Node todo = last.next;</pre> | <p>F: Loop Invariant $_0$</p>  | <p>K: Give the complete code</p> <pre>void ReverseCycle(Node last) { Node done = last; Node todo = last.next; while(todo.next!=last) { Node temp = todo.next; todo.next = done; done = todo; todo = temp; } }</pre> |
| <p>C: Loop Invariant $_t$</p>  | <p>E: Code C to D</p> <pre>Node temp = todo.next; todo.next = done; done = todo; todo = temp;</pre>  | <p>D: Loop Invariant $_{t+1}$</p>  | |
| <p>H: Loop Invariant $_n$</p>  | <p>I: Exit Condition</p> <p>"done==last" is good but does not differentiate from F if(todo.next==last) exit;</p> <p>J: Code H to B</p> <p>Nothing to do except return which will deallocate the memory for done and todo</p> | <p>B: Post Condition</p>  | |

10. (Answer in slides)

Lake Problem:

You are in the middle of a lake of radius one. You can swim at a speed of one and can run infinitely fast. There is a smart monster on the shore who can't go in the water but can run at a speed of four. Your goal is to swim to shore arriving at a spot where the monster is not and then run away. If you swim directly to shore, it will take you 1 time unit. In this time, the monster will run the distance $\pi < 4$ around to where you land and eat you. Your better strategy is to maintain the most obvious loop invariant while increasing the most obvious measure of progress for as long as possible and then swim for it. Describe how this works.

- Answer: I am going to complete the iterative algorithm steps in the order in which I thought of them. The disadvantage of this order is that steps that depend on later steps are not completely

complete. But hopefully if you read the solution a second time you will see that each of these steps can be completed fitting them nicely together.

- Define Loop Invariant:

The loop invariant that you would like to maintain is that you are diametrically opposed to the monster, i.e. the line between him and you goes through the center.

I would say this is a *Fairy Godmother* type of loop invariant. I have arrived from Mars, some amount of progress has been made already and my Fairy God Mother has set the state of the computation to be just what I want so that I feel comfortable. OK, maybe I would feel more comfortable if I was safely in my bed but this as comfortable as I could be given the amount of progress that has been made.

- Establishing the Loop Invariant $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$:
Initially, the loop invariant is trivially true because you are at the center.
- Maintaining the Loop Invariant $\langle loop-invariant' \rangle \ \& \ not \ \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$:

Your initial goal is simply to maintain the loop invariant as long as you can. You keep an eye on the monster. If he runs clockwise on shore, then you swim counter clockwise that the rate necessary to maintain that you are diametrically opposed to the monster. If he suddenly switches to running counter clockwise, then you switch your direction too. If you are currently near the shore, then you will have to swim almost as fast as he runs to accomplish this. The problem is that he runs four times faster than you can swim. On the other hand, if you are at the center, then you only need to tread water to accomplish this initial goal. The dividing line between these two extremes is when your current location is at radius $r \leq \frac{1}{4}$ from the center. In such a case, you are able to swim around your circle of radius r at least as fast as he can run around the lake, even though he swims four times faster because your circle has at most $\frac{1}{4}$ the circumference of his. Hence, as long as $r \leq \frac{1}{4}$, you have no problem maintaining the loop invariant.

- Making Progress:

Your measure progress is your distance r from the center. Your movements for maintaining the loop invariant move you parallel to the shore and hence do not make you any progress according to this measure. However, if r is strictly less than $\frac{1}{4}$, then you do not need all of your speed to maintain the loop invariant. Use your extra speed to make progress by swimming outward.

- Define the Step. As said, some of your swimming effort is swimming in the circle perpendicular to the shore in order to keep the center of the lake between you and the monster and your remaining effort is making progress swimming towards the nearest shore. Instead of swimming in a stair case alternating between these two goals you can use linear algebra to combine these two vectors into one vector that moves you in a smooth spiral way from the center. Of course the direction of your spiral may keep changing as the monster gets frustrated and changes his direction.
- Define Exit Condition: As said in the iterative algorithm steps, one possible reason for exiting the loop is that you are no longer able to maintain the loop invariant. In this case, we fail to make progress when $r \geq \frac{1}{4}$ circle. Hence, we make this our exit condition.

- Ensuring that we actually Exit:

The official rules of loop invariant algorithms is that you must make progress of at least one each time step because steps of size $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots$ will get you close but never quite there.

Years ago I checked the converging issue and my memory is if you swimming in a stair case alternating between swimming perpendicularly and towards shore, then the amount you have left to swim towards shore is not enough to get you to ever reach $r = \frac{1}{4}$ in a finite amount of time, i.e. you will get closer and closer but never quite reach it. However, if you use linear algebra to combine these two vectors into one vector that spirals, then you do in fact converge to $r = \frac{1}{4}$ in a finite time. Check this if you like.

To avoid the converging issue, lets set some small number ϵ and exit when our distance from the center is $r \geq \frac{1}{4} - \epsilon$. We know that we will eventually meet this exit condition because

as long as $r < \frac{1}{4} - \epsilon$, in each “time unit” we are guaranteed to make some constant amount $\mathcal{O}(\epsilon)$ amount of progress.

- Obtaining the postcondition $\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$:

By the exit condition $r \geq \frac{1}{4} - \epsilon$. By the loop invariant, the monster is on the opposite shore from your current location. The post loop code is for you to swim as fast as you can directly for shore. This take you $1 - r \approx \frac{3}{4}$ time units. During this time the monster can run a distance of 3. However, because he is diametrically opposed to you when you start your sprint, he needs to run a distance of $\Pi > 3$, which is too far for him. You then run to safety meeting the post condition!

In contrast, here is a wrong answer.

- 1) Basic steps: Swim in the opposite direction of monster’s current location.
- 2) Loop invariant: You’ve been swimming to the shore while your head is in the opposite direction of monster’s current location.
 - * Being a picky marker, I will complain that this sounds more like a description of the algorithm than like a picture of the current state of the computation. I hope this wrong answer help to demonstrate the danger of this. Though this loop invariant does tell you what you have been *doing*, it does not tell you what you need to know, which is where you *currently* are wrt the monster.
- 3) Main steps: Look around and check monster’s location while you keep swimming and change the direction if it’s needed to keep your direction opposite to monster’s location.
 - * (Fancy talk for “Panic and swim away from the monster.” Certainly this is the most natural thing for a frightened person to do. And I am not trying to prove here that this algorithm does not work (it may). I am trying to prove that this proof that this algorithm works is a faulty proof.
- 4) Measure of progress: distance to the shore is the measure of progress.
 - * For sure, this is the right measure of progress.
- 5) Make progress: It makes progress because where ever monster is at shore, if you keep swimming to the shore at the opposite direction of monster’s location. You get closer to the shore, which mean the distance to the shore is reduced.
 - * But what if he is on the shore you are closest too. Then your algorithm swims you away from shore and you fail to make progress - in fact, just the opposite. You could well swim in circles for ever.
- 6) Maintain loop variant: You have been swimming in the opposite direction of monster’s location from the previous loop. Loop invariant is maintained by checking the monster’s location and keeping your head in the opposite direction of monster’s location while you’re swimming.
 - * True, you do manage to keep doing this same algorithm and hence because your loop invariant only states that you keep doing this same algorithm, you do in fact maintain the loop invariant (for what ever that is worth).
- 7) Establishing loop invariant: You obtain loop invariant by checking the monster’s direction and starting to swim to the opposite direction of monster’s location.
 - * Certainly a good start.
- 8) Exit condition: distance to the shore is zero.
 - * The problem is that this exit condition might never be met.
- 9) Ending: By the exit condition, you are at the shore and by the loop invariant your head is is pointed in the opposite direction of monster’s location. The post condition follows.
 - * Your head my well be pointed in the opposite direction of monster’s location, but because your loop invariant says nothing about where the monster is wrt your current location, he may well be standing right beside you. On the other hand, it may be a good thing that your head is facing in the opposite direction because then you might aware of moment that he eats you.

11. Iterative 3 In A Row In An Infinite Line:

The game board consists of a single line of squares that is infinitely long in both directions. Each iteration your opponent places a white piece and then you place a black piece. If a player gets three in a row he wins. Your primary goal is to stop him from doing this. If the game goes on forever then this is considered a tie.

Your game strategy will be to maintain the following loop invariants

After we each have played the same number of times the following are true.

- (a) Every contiguous block of whites has a black on its left end.
- (b) Every contiguous block of whites of length two or more also has a black on its right end.
- (c) There are no contiguous blocks of white of length three or more.

(i.e. A block consisting of one white has a black to its left, a block with two whites has a black on both ends, and there are no blocks of length three or more.)

The exit condition will be that someone has won.

The post condition will be that you win.

Your task is to follow all the loop invariant steps outlined in the steps and then to prove that either the game goes on forever as a tie or you win.

Suppose your opponent has just placed a white. There are nine cases depending on whether there is a black, white, or blank on the left of this newly placed white and whether there is a black, white, or blank on the right. To help you, I group these nine into the five cases indicated in the following five lines. Then I group these further into three cases indicated by the black dots. In each of these five cases, state where you will place your black in response and in each of these three cases, prove that this maintains each of the three loop invariants.

- There is a blank to the left of where he placed his white.
There is a black to the left.
- There is a white to the left and a blank to the right.
There is a white to the left and a black to the right.
- There is a white to the left and a white to the right.

Is it possible to have an algorithm that guarantees you win even if you go first?

- Answer:

Establishing the Loop Invariant: $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$

The precondition gives that the board is empty. A very important but silly fact is if there are no elephants in the room, then the statement “Every elephant in the room is pink” is a true statement. Similarly, if there are no contiguous block of whites, then L1=“Every contiguous block of whites has a black on its left end.” is trivially true. Similarly L2. L3, which state that there are no contiguous blocks of white of length three or more, is clearly true when the board is empty.

Responding Move and Maintaining the Loop Invariant: $\langle loop-invariant' \rangle \ \& \ not \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$.

When he places a white, I move as follows.

- If there is a blank to the left I go there.

If there is a black to the left I go somewhere else trying to win.

- * We made sure that this block of whites has a black on its left end. Hence, we maintained LI1.
- * Because there is not a white to the left of where he went, he did not extend a white block further to the right. Neither did he extend a white block further to the left because LI1 ensures that at the beginning of his turn every white block has a black on its left end. Hence his white does not make a white block longer and hence maintains LI2 and LI3.

- If there is a white to the left and a blank to the right I go in the blank.
If there is a white to the left and a black to the right I go somewhere else trying to win.
- * By LI1 this block of whites already has a black on the left end so LI1 is maintained.
- * We made sure that this block of whites has a black on its right end. Hence, we maintained LI2.
- * The opponent could not have made a block of two (or more) whites longer because by LI2 such a block would have had a black on both ends. Hence, LI3 is maintained.
- It is imposable that there is a white to the left and a white to the right.
- * LI1 ensures that at the beginning of the opponents turn, there is never a blank to the left of a white. Hence, there could not have been such a blank space for him to go into.

Obtaining the Postcondition: $\langle loop\text{-invariant} \rangle \ \& \ \langle exit\text{-cond} \rangle \ \& \ code_{post\text{-loop}} \Rightarrow \langle post\text{-cond} \rangle$:

By the loop invariant L3, we know that the opponent has not won.

By the exit condition, someone has won.

Hence it must be us.

Either the game go on forever as a tie or you win We don't have a measure of progress, because the game is likely to go on forever as a tie. But we did prove that if the game does exit, you win.

Going First: This same thing proves that there is no algorithm that guarantees you win even if you go first.

Unfair Advantage: Now suppose you are white, i.e. trying to grow the long block. Suppose that you get 100 times as many turns. You could then build a contiguous block of length 200, i.e. you put a 100, he caps one end by placing a black there, you place a second 100 on the other end, and he caps that end with a black. If you did manage to build a really long block, it would take you a while. Then in just two moves he can destroy it by capping both ends. A fun question is to what is the longest contiguous block of whites that you can build? To make the algorithm easier, lets assume that the opponent is not trying to build a block of blacks. His only goal is to prevent you from building a long contiguous block of white by capping the ends of any block you are working on with a black.

12. Iterative n In A Row In An Infinite Line:

The game board is an infinitely long line of squares. In a normal round, you will place a white and then your opponent a black. The goal is to build long contiguous blocks of your colour. We proved in test 1 that there is no algorithm that allows you to build a block longer than two. We will now develop an algorithm that builds arbitrarily long contiguous white blocks. To help, you will be given the advantage that every ten round you get to place a white but your opponent must skip his turn. (The regular expression is $((WB)^9W)^*$.)

It seemed at first to me that even if you got 100 times as many turns, that you could not build a contiguous block longer than 200, i.e. you put a 100, he caps one end by placing a black there, you place a second 100 on the other end, and he caps that end with a black. If you did manage to build a really long block, it would take you a while. Then in just two moves he can destroy it by capping both ends. Surprisingly, you can build arbitrarily long lines. (Funny though you can't just build a block that gets longer and longer arbitrarily. You must have the length you want to build in mind when you start.)

To make the algorithm easier, lets assume that the opponent is not trying to build a block of blacks. His only goal is to prevent you from building a long contiguous block of white by capping the ends of any block you are working on with a black. All blocks that we start will be far enough away from each other that they will never grow long enough to interact with each other. We will be cautious. If the opponent caps one of the ends of one of our blocks or even puts a black anywhere near that block, then we will not wait for him to cap the second end but will abandon that block completely. Hence, all we are concerned about is how many blocks we have that have no blacks near them and how long are these blocks are. The adversary each of his turns, being as greedy as he can be, will simply cap

the first end of our currently longest such block. Given this, we will keep our blocks as close to being the same size as possible.

Input: The input to this problem is an integer n (say a billion).

Precondition: The board is empty.

Postcondition: You have produced a continuous block of whites of length n .

Iterations: This will be an iterative algorithm. You will have “iterations” for $i = 0, 1, \dots, n$. Each such iteration will contain as many rounds of turns as you need to make progress and maintain the loop invariant.

Loop Invariant: After i “iterations”, you have constructed many *special* blocks.

Length: Each such special block consists of i whites in a row.

Isolation: Each is far away from any black and from each other.

Number: The number of such special blocks is 10^{n-i} .

In addition to these special blocks the board will contain many white blocks that are abandoned because they have a black on one end.

- Complete Jeff’s steps in completing the description/proof of this algorithm.
- Also compute the total number N of whites that you place.

- Answer:

Establishing the Loop Invariant: $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$

The precondition gives that the board is empty and perhaps the value n . To establish the loop invariant for $i = 0$, we must produce 10^n blocks of whites of length zero far away from each other and far away from any black. This requires placing $N_0 = 0$ whites. It only requires deciding where the blocks will be.

Maintaining the Loop Invariant: $\langle loop-invariant' \rangle \ \& \ not \ \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$.

Suppose the LI is true for $i-1$ “iterations”, i.e. the board contains 10^{n-i+1} blocks of whites each of length $i-1$ far away from each other and far away from any black. Our goal is to make these blocks one longer. For each of our next 10^{n-i+1} turns, we simply place a white next to one of the blocks of these length $i-1$ making it a block of length i . Our opponent gets nine moves for each of our ten. Hence, during these 10^{n-i+1} turns of ours he gets $0.9 \times 10^{n-i+1}$ turns. During each of these moves he can, if he wants, “destroy” one of our blocks of length i by placing a black near it. But whatever he does, he cannot destroy more than $0.9 \times 10^{n-i+1}$ of them. As required by the loop invariant, this leaves $10^{n-i+1} - 0.9 \times 10^{n-i+1} = 10^{n-i}$ blocks of length i . During this i^{th} iteration we placed $N_i = 10^{n-i+1}$ white stones.

Obtaining the Postcondition: $\langle loop-invariant \rangle \ \& \ \langle exit-cond \rangle \ \& \ code_{post-loop} \Rightarrow \langle post-cond \rangle$:

The exit condition is that $i = n$. By the loop invariant there is $10^{n-n} = 1$ contiguous white block of length n .

Measure of Progress: Clearly the measure of progress is the value i , it increases each iteration and after n such iteration the exit condition is obtained.

Running Time N : The total number of whites that you place is $N = \sum_{i=1}^n N_i = \sum_{i=1}^n 10^{n-i+1} \leq 2 \cdot 10^n$.

13. **Multiplying:** The ancient Egyptians and the Ethiopians had advanced mathematics. Merely by halving and doubling, they could multiply any two numbers correctly. Say they want to buy 15 sheep at 13 Ethiopian dollars each. Here is how he figures out the product. Put 13 in a left column, 15 on the right. Halve the left value; you get $6\frac{1}{2}$. Ignore the $\frac{1}{2}$. Double the right value. Repeat this (keeping all intermediate values) until the left value is 1. What you have is

| | |
|----|-----|
| 13 | 15 |
| 6 | 30 |
| 3 | 60 |
| 1 | 120 |

Even numbers in the left column are evil and, according to the story, must be destroyed, along with their guilty partners. So scratch out the 6 and its partner 30. Now add the right column giving $15 + 60 + 120 = 195$, which is the correct answer.

Give the key steps to describe this LI algorithm.

Hint: Our code does not need remember all previous values. Instead, let x and y hold the original input integers, let ℓ hold the current left value, r the current right, and s the sum of all previous right values that will be included in the final answer.

Hint: Break steps within the loop (and proof that the loop invariant is maintained) into two steps. In the first step, if ℓ is odd it decreases by one. In the second step, ℓ (now even) is divided by two. These steps, must update r and s as needed. The running values will be as follows.

| ℓ | r | s |
|--------|-----|-----|
| 13 | 15 | 0 |
| 12 | 15 | 15 |
| 6 | 30 | 15 |
| 3 | 60 | 15 |
| 2 | 60 | 75 |
| 1 | 120 | 75 |
| 0 | 120 | 195 |

(a) **Specification:**

$\langle pre-cond \rangle$ is that the input instance consists of two positive integers x and y .

$\langle post-cond \rangle$ is that the output is their product.

(b) * **Define Loop Invariant:** Give a meaningful loop invariant relating the current values of ℓ , r , s , x , and y . (Hint, look at the Shrinking Instance loop invariant.) (5 marks)

- Answer: The loop invariant is $\ell \times r + s = x \times y$.

(c) * **Establishing the Loop Invariant:** (5 marks)

- Answer: $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$:
 - $\langle pre-cond \rangle$ assures us that we have two integers x and y .
 - The code before the loop sets $\ell = x$, $r = y$, and $s = 0$.
 - This trivially establishes the loop invariant by giving $\ell \times r + s = x \times y + 0$.

(d) **Picture:** If it is helpful for you, draw pictures to give a geometric explanation for the steps. (0 marks)

- Answer: Draw an ℓ by r rectangle and an area of s on the side. The first step consists of removing one row (with area $1 \times r$) from the rectangle and moving that area to the side. The second step consists of cutting the rectangle in half top to bottom and moving the top half to the right side of the bottom half. Both of these steps keep the total area the same.

(e) **Step:** If it is helpful for you, define the Step in the loop (or even all the code). (0 marks)

- Answer:


```

algorithm EthiopianMultipling ( $x, y$ )
   $\langle pre-cond \rangle$ :  $x$  and  $y$  are positive integers.
   $\langle post-cond \rangle$ : Outputs  $x \times y$ .
  begin
     $\ell = x, r = y, s = 0$ 
    loop
       $\langle loop-invariant \rangle$ :  $\ell \times r + s = x \times y$ .

      exit when  $\ell = 0$ 
      if ( $\ell$  is odd) then
         $\ell = \ell - 1$ 
         $s = s + r$ 
      end if

       $\langle loop-invariant \rangle$ :  $\ell \times r + s = x \times y$ .
    end loop
  end
      
```

```

     $\ell = \ell/2$ 
     $r = 2 \times r$ 
end loop
return( $s$ )
end algorithm

```

(f) Maintaining the Loop Invariant for the first half of the step. (5 marks)

- Answer: $\langle \text{loop-invariant}' \rangle \ \& \ \text{not} \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{loop1}} \Rightarrow \langle \text{loop-invariant}'' \rangle$:
Let ℓ', r', s' be the values when at the top of the loop and ℓ'', r'', s'' be the values after the first step.
 - $\langle \text{loop-invariant}' \rangle$ gives $\ell' \times r' + s' = x \times y$.
 - $\text{code}_{\text{loop1}}$ (if ℓ' is even) gives $\ell'' = \ell'$ and $s'' = s'$, in which case $\langle \text{loop-invariant}'' \rangle$ is obvious.
 - $\text{code}_{\text{loop1}}$ (if ℓ' is odd) gives $\ell'' = \ell' - 1$ and $s'' = s' + r'$.
 - We prove $\langle \text{loop-invariant}'' \rangle$ as follows: $\ell'' \times r'' + s'' = (\ell' - 1) \times r' + (s' + r') = \ell' \times r' + s'$, which by the loop invariant is $x \times y$.

(g) Maintaining the Loop Invariant for the second half of the step. (5 marks)

- Answer: $\langle \text{loop-invariant}'' \rangle \ \& \ \text{not} \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{loop2}} \Rightarrow \langle \text{loop-invariant}''' \rangle$:
Let ℓ'', r'', s'' be the values after the first step and let ℓ''', r''', s''' be the values when coming around to the top of the loop again.
 - $\langle \text{loop-invariant}'' \rangle$ gives $\ell'' \times r'' + s'' = x \times y$.
 - $\text{code}_{\text{loop2}}$ gives $\ell''' = \ell''/2$ and $r''' = 2r''$.
 - We prove $\langle \text{loop-invariant}''' \rangle$ as follows: $\ell''' \times r''' + s''' = (\ell''/2) \times (2r'') + s'' = \ell'' \times r'' + s''$, which by the loop invariant is $x \times y$.

(h) * **Ending:** Obtain the Postcondition.

Hint: What is the Ethiopian exit condition? How might you improve on this? (5 marks)

- Answer: The Ethiopians exit when $\ell = 1$. But this being odd, they must add r to s . We will iterate one more time and exit when $\ell = 0$.
 $\langle \text{loop-invariant} \rangle \ \& \ \langle \text{exit-cond} \rangle \ \& \ \text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$:
 - $\langle \text{loop-invariant} \rangle$ gives $\ell \times r + s = x \times y$
 - $\langle \text{exit-cond} \rangle$ gives $\ell = 0$.
 - This gives $s = 0 \times r + s = \ell \times r + s$, which by the loop invariant is $x \times y$.
 - $\text{code}_{\text{post-loop}}$ returns s .
 - This proves $\langle \text{post-cond} \rangle$ which is that $x \times y$ is returned.

(i) **Time Complexity:** Give the time complexity of our multiplication algorithm. (5 marks)

- Answer: The “size” of the input is $n = \log_2 x + \log_2 y$, which is the number of bits to write down x and y . (Alternatively suppose that the input instance x and y are each n bit numbers.)

Because x comes down by a factor of 2 each iteration, the number of iterations is $\log_2 x$. (Extra: The measure of progress can be the value of ℓ or even more elegantly $\log_2 \ell$. Progress is made each iteration because the value of ℓ is cut in half and hence $\log_2 \ell$ goes down by one. Initially, the measure is x (or $\log_2 x$) and the algorithm halts when they are zero.)

Each iteration needs to do an addition which required $\Theta(n)$ bit operations. (Note no value increases beyond $x \times y = 2^n \times 2^n = 2^{2n}$ and hence is at most a $2n$ bits.)

Hence, the total algorithm uses at most $\Theta(n^2)$ bit operations, which is quadratic in the size $2n$ of the input. This time equivalent to that needed for the high school algorithm for multiplying.

Not asked for: If pebbles are used then eventually the answer $x \times y$ is counted out in pebbles. This alone requires $\Omega(x \times y) = \Omega(2^{2n})$ pebble operations. This is exponential in the input

size $2n$. But if the input is given to us in pebbles, it might be more fair to call the “size” the number of pebbles, which is $x + y$. In which, case the time $x \times y$ is also quadratic in the input size. Note this is equivalent in time to lay out a rectangle of x by y pebbles.

(j) **A Comparison:** (Not asked for) This algorithm seems very strange. Compare it to using the high school algorithm for multiplying in binary.

- Answer: Though this algorithm seems strange, it is in fact exactly the same as the high school algorithm for multiplying in binary. Lets us first understand multiplication $x \times y$ in terms of bits. Let $x_4x_3x_2x_1x_0$ denote the bits of x written in binary. For example $x = 58 = 11001_2$. The bit x_i is shifted into place by multiplying it by 2^i . This gives $x = \sum_i 2^i x_i$. Plugging this in gives that $x \times y = [\sum_i 2^i x_i] \times y = \sum_i [2^i x_i y]$.

$$\begin{array}{r}
 Y \quad 111010 \\
 X \quad \underline{101} \\
 \quad 111010 \quad 2^0 a \text{ Y} \\
 \quad 000000 \quad 2^1 a \text{ Y} \\
 \quad 111010 \quad 2^2 a \text{ Y} \\
 \hline
 100100010
 \end{array}$$

Now lets see how the high school algorithm for multiplying in binary implements this. First the bits of y are written on the top line and the bits of x , namely $x_4x_3x_2x_1x_0$, are written below it. On the i^{th} line below this, we multiply x_i by y and shift this over i spots. This gives $2^i y x_i$ on this i^{th} line. The algorithm then adds these lines up giving the answer $\sum_i 2^i y x_i$ as required.

Now lets consider our algorithm. Lets first see how it determines the bits of x . The variable ℓ starts at x and comes down by a factor of two rounded down each iteration and hence after i iterations is $\ell_i = \lfloor \frac{x}{2^i} \rfloor$. Similarly $r_i = 2^i y$. An iteration adds in r_i to s iff ℓ_i is odd. $\ell_i = \lfloor \frac{x}{2^i} \rfloor$ is odd iff the i^{th} bit of x is $x_i = 1$. For example, $x = 13$ in binary is $x = 1101_2$. Here, $x_0 = 1$ because $\ell_0 = x = 13$ is odd and $x_1 = 0$ because $\ell_1 = \lfloor \frac{x}{2} \rfloor = 6$ is even. Hence, the bit x_i indicates whether or not r_i is added to s . Hence, our answer, which is the final value of s , is $\sum_i r_i x_i = \sum_i 2^i y x_i$ as required.

14. (Answer in slides)

Multiplying using Adding: My son and I wrote a JAVA compiler for his grade 10 project. We needed the following. Suppose you want to multiply $X \times Y$ two n bit positive integers X and Y . The challenge is that the only operations you have are adding, subtracting, and tests like \leq . The standard high school algorithm seems to require looking at the bits of X and Y and hence can't obviously be implemented here. The Ethiopian algorithm requires dividing by two, so can't be implemented either. A few years after developing this algorithm, I noticed that it is in fact identical to this high school multiplication algorithm, but I don't want to tell you this because I don't want you thinking about the algorithm as a whole. This will only frighten you. All you need to do is to take it one step at a time. I want only want you to establish and maintain the loop invariant and use it to get the post condition. (The flavor is very similar to what was done in the Ethiopian problem on the practice test.) To help, I will give you the loop invariants, the measure of progress, and the exit condition. I also want you to explain what the time complexity of this algorithm is, i.e. the number of iterations and the total number of bit operations as a function of the size of the input.

We have values $i, x, a, u[]$, and $v[]$ such that

Useful Arrays: Before the main loop, I will set up two arrays for you with the following values. Then they will not be changed.

LI0': $u[j] = 2^j$, (for all j until $u[j] > X$)

LI0'': $v[j] = u[j] \times Y$. Note $v[j] - u[j] \times Y = 0$

My Code: For completion, I include my code, but it is not necessary for you to understand it.

$$\begin{array}{l}
 u[0] = 1 \\
 v[0] = Y
 \end{array}$$

```

j = 0
while( u[j] ≤ X )
    u[j + 1] = u[j] + u[j]
    v[j + 1] = v[j] + v[j]
    j = j + 1
end while

```

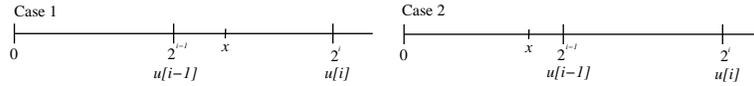
For Main Loop: These are the loop invariants for you to deal with.

LI0: Neither X nor Y change.

LI1: $X \times Y = x \times Y + a$ (i.e. Shrinking Instance)

LI2: $x \geq 0$

LI3: $x < 2^i = u[i]$ (below are two cases)



Measure of progress: i . It decreases by 1 each iteration.

Exit Condition: $i = 0$

Use the steps laid out in class to complete the description of the algorithm.

- Answer:

Establishing the Loop Invariant $\langle pre-cond \rangle \ \& \ code_{pre-loop} \Rightarrow \langle loop-invariant \rangle$:

$code_{pre-loop}$:

$i = j$

$x = X$

$a = 0$

LI1: $x \times Y + a = X \times Y + 0 = X \times Y$.

LI2: $x = X \geq 0$ by pre-condition

LI3: $x = X < u[j] = u[i]$ by exit condition of while loop that constructs $u[j]$.

Steps:

```

loop
     $\langle loop-invariant \rangle$ 
    exit when  $i = 0$ 
     $i = i - 1$ ;
    if( $u[i] \leq x$ )
         $x = x - u[i]$ 
         $a = a + v[i]$ 
    end if
end loop

```

Maintaining the Loop Invariant $\langle loop-invariant' \rangle \ \& \ not \ \langle exit-cond \rangle \ \& \ code_{loop} \Rightarrow \langle loop-invariant'' \rangle$:

Let x' be the value of x when we are at the top of the loop and let x'' be that after going around the loop one more time. (Same for i and a)

There are two cases:

if($u[i''] \leq x'$): The code in the step gives $i'' = i' - 1$, $x'' = x' - u[i'']$, and $a'' = a' + v[i'']$.

Maintain LI1: $x'' \times Y + a'' = (x' - u[i'']) \times Y + (a' + v[i'']) = x' \times Y - u[i''] \times Y + a' + v[i''] = (x' \times Y + a') + (v[i''] - u[i''] \times Y) = (X' \times Y \text{ by LI1}) + (0 \text{ by LI0}) = X \times Y$.

Maintain LI2: By the if statement $u[i''] \leq x'$ and hence $x'' = x' - u[i''] \geq 0$.

Maintain LI3: $x'' = x' - u[i''] = x' - 2^{i'-1} < 2^{i'} - 2^{i'-1}$ (by LI3) $= 2^{i'-1} = 2^{i''} = u[i'']$.

else The code in the step gives $i'' = i' - 1$, $x'' = x'$, and $a'' = a'$

Maintain LI1&2: Trivial

Maintain LI3: By the else of the if statement, $u[i''] > x'$.

Obtaining the postcondition $\langle \text{loop-invariant} \rangle$ & $\langle \text{exit-cond} \rangle$ & $\text{code}_{\text{post-loop}} \Rightarrow \langle \text{post-cond} \rangle$:

By exit condition, $i = 0$.

By LI2&3, $0 \leq x < u[i] = 1$, and hence $x = 0$.

By LI1, $X \times Y = x \times Y + a = 0 \times Y + a = a$.

It follows that the postcondition is met simply by outputting a .

Time Complexity: The “size” of the input is $n = \log_2 X + \log_2 Y$, which is the number of bits to write down X and Y . The number of iterations is the initial value of i , which is set so that $X = x < u[i] = 2^i$, giving $i = \lceil \log_2 X \rceil \leq \mathcal{O}(n)$. Each iteration requires add/subtract operations which take $\mathcal{O}(n)$ bit operations. Hence this algorithm will require $\mathcal{O}(n^2)$ bit operations.

A Comparison: Though this algorithm seems strange, it is in fact exactly the same as the high school algorithm for multiplying in binary (and hence the same as the Ethiopian algorithm). Lets us first understand multiplication $X \times Y$ in terms of bits. Let $X_4X_3X_2X_1X_0$ denote the bits of X written in binary. For example $X = 58 = 11001_2$. The bit X_i is shifted into place by multiplying it by 2^i . This gives $X = \sum_i 2^i X_i$. Plugging this in gives that $X \times Y = [\sum_i 2^i X_i] \times Y = \sum_i [2^i X_i Y]$.

$$\begin{array}{r}
 Y \quad 111010 \\
 X \quad \quad 101 \\
 \hline
 111010 \quad 2^0 a \text{ Y} \\
 000000 \quad 2^1 a \text{ Y} \\
 111010 \quad 2^2 a \text{ Y} \\
 \hline
 100100010
 \end{array}$$

Now lets see how the high school algorithm for multiplying in binary implements this. First the bits of y are written on the top line and the bits of X , namely $X_4X_3X_2X_1X_0$, are written below it. On the i^{th} line below this, we multiply X_i by Y and shift this over i spots. This gives $2^i Y X_i$ on this i^{th} line. The algorithm then adds these lines up giving the answer $\sum_i 2^i Y X_i$ as required.

Now lets consider our algorithm. Lets first see how it determines the bits of X . Let x_i denote the value of x in the iteration in which i is i . Our first value of i is the number of bits in X , here 5, because it is the smallest for which $X = x < u[i] = 2^i = 100000_2$. A loop invariant that we will maintain is that x_i will always be the integer formed from the right most i bits $X_{i-1} \dots X_2X_1X_0$ of $X = 11001_2$, namely $x_5 = 11001_2$, $x_4 = 1001_2$, $x_3 = 001_2$, $x_2 = 01_2$, $x_1 = 1_2$, and $x_0 = 0_2$. We establish this loop invariant by setting $x_5 = x_i = X$. We set $u[i-1] = 10000_2$ so when written in binary it is zero everywhere except a 1 in the location of the bit X_{i-1} . The iteration tests whether $u[i-1] \leq x_i$ which tells us whether or not $X_{i-1} = 1$. For example, we know that $X_4 = 1$ because $u[4] = 10000 \leq 11001_2 = x_4$ we know that $X_2 = 0$ because $u[2] = 100 > 001_2 = x_3$. We maintain this loop invariant about x_i being the right i bits of X by removing its left most bit using $x_{i-1} = x_i - u[i-1]$. For example, $x_4 = x_5 - u[4] = 11001_2 - 10000_2 = 1001_2$ and $x_2 = x_3 = 001_2 = 01_2$. Each iteration also adds $v[i-1]$ into a iff $u[i-1] \leq x_i$ and hence iff $X_{i-1} = 1$. Hence, our answer, which is the final value of a , is $\sum_i v[i] X_i = \sum_i 2^i Y X_i$ as required.

15. Computing square root digit by digit: The following is the algorithm for taking square roots that I learned in high school. The loop invariant is both the classic more of the input and the classic more of the output. As such, you should be able to quickly complete the steps. Do not panic. Your job is simply to trust and follow the steps. It should all fall out simply.

I wrote a long and fascinating explanation of how this algorithm looks like long division. But on reflection it was not necessary to do the test and only distracted. It will be included in the solution set.

As I have often said — do not try to understand this algorithm.

algorithm *SquareRoot*(p)

<pre-cond> P is an integer

but we pair its digits as $P = \langle a_1, \dots, a_t, \dots, a_n \rangle = \langle 07, 80, 14, 71, 23 \rangle = 0780147123$.

<post-cond> Q is the square root of P rounded down to the nearest integer.

We consider its digits as $Q = \langle b_1, \dots, b_t, \dots, b_n \rangle = \langle 2, 7, 9, 3, 1 \rangle = 27931$.

$R = P - Q^2$ is the remainder.

begin

??

loop

<loop-invariant>: From more of the input we have $p = \langle a_1, \dots, a_t \rangle$.

From more of the output we have $q = \langle b_1, \dots, b_t \rangle$.

As integers, q is the largest integer for which $r = p - q^2$ and $r \geq 0$

exit when ??

$a = a_{t+1}$ = next pair of digits in P

$p = 100p + a$

$R = 100r + a$

loop $b = 9, 8, \dots, 0$

$s = (10(2q) + b) \times b$

$r = R - s$

exit when $r \geq 0$

end loop

$q = 10q + b$

end loop

return($\langle q, r \rangle$)

end algorithm

Hint: Ignore the loop over b . Just assume that it is setting b to be the biggest value in $\{0, \dots, 9\}$ for which r is non-negative.

(a) ($8 + 8 + 24 = 40$ marks) Complete the three key loop invariant steps for this algorithm.

- Answer:

Establishing the Loop Invariant *<pre-cond>* & *code_{pre-loop}* \Rightarrow *<loop-invariant>*:

Initially $t = 0$ pairs of digits have been read. Having no digits, both p and q are 0. The initial code sets $p = 0$, $q = 0$, and $r = 0$. The loop invariant $p = q^2 + r$ follows trivially with $0 = 0 - 0^2 \geq 0$.

Obtaining the postcondition

<loop-invariant> & *<exit-cond>* & *code_{post-loop}* \Rightarrow *<post-cond>*:

<exit-cond>: The exit condition will be that $t = n = 5$ and hence all of the pairs of digits of P have been included.

<loop-invariant>: At this time $p = \langle a_1, \dots, a_n \rangle = \langle 07, 80, 14, 71, 23 \rangle = 780147123 = P$ is the full input.

code_{post-loop}: The final code sets $Q = q$ and $R = r$ and returns them.

<loop-invariant>: Together with the previous statements gives that Q is the largest integer for which the remainder $R = P - Q^2 \geq 0$.

<post-cond>: It is not too hard to see that this makes Q is the square root of P rounded down to the nearest integer.

More specifically we will have that $q_t = \langle b_1, b_t, \dots, b_n \rangle = \langle 27931 \rangle = 27931$ and the remainder is $r_t = 780147123 - 27931^2 = 6263$.

Maintaining the Loop Invariant

$\langle \text{loop-invariant} \rangle_t$ & not $\langle \text{exit-cond} \rangle$ & $\text{code}_{\text{loop}} \Rightarrow \langle \text{loop-invariant} \rangle_{t+1}$:

Let p_t , q_t and r_t be the value of our variables when we are at the top of the loop and let p_{t+1} , q_{t+1} and r_{t+1} be that after going around the loop one more time.

It is easy to see that the first two loop invariants are maintained.

$\langle \text{loop-invariant} \rangle_t$: We have for the old values that $p_t = \langle a_1, \dots, a_t \rangle$ and $q_t = \langle b_1, \dots, b_t \rangle$.

$\langle \text{loop-invariant} \rangle_{t+1}$: What we need for the new values is that $p_{t+1} = \langle a_1, \dots, a_t, a_{t+1} \rangle$ and $q_{t+1} = \langle b_1, \dots, b_t, b_{t+1} \rangle$. In terms of digits this just requires concatenating on the new digits a_{t+1} and b_{t+1} .

Now let's consider p and q as integers.

Example: If $P = 0780147123$ and $t = 2$, then $p_t = \langle 07, 80 \rangle = 0780$, $a_{t+1} = 14$, and $p_{t+1} = \langle 07, 80, 14 \rangle = 078014$.

$\text{code}_{\text{loop}}$: The code gives the correct relationship that $p_{t+1} = 100p_t + a_{t+1}$ and $q_{t+1} = 10q_t + b_{t+1}$.

The third loop invariant $r = p - q^2 \geq 0$ requires more math.

$\langle \text{loop-invariant} \rangle_t$: By the loop invariant_t, (about old values)

$$r_t = p_t - q_t^2$$

$\text{code}_{\text{loop}}$: By the code

$$p_{t+1} = 100p_t + a_{t+1}$$

$$q_{t+1} = 10q_t + b_{t+1}$$

$$R_{t+1} = 100r_t + a_{t+1}$$

As done in long division the new digits a_{t+1} are brought down and tacked on to the digits of the current remainder.

$$s_{t+1} = [10(2q_t) + b_{t+1}] \times b_{t+1}$$

Think of $2q_t$ as a sequence of digits and concatenating the new digit b_{t+1} onto its right.

Then multiply this by b_{t+1} .

$$r_{t+1} = R_{t+1} - s_{t+1}$$

$$b_{t+1} = \text{largest digit so that } r_{t+1} \geq 0$$

Note $r_{t+1} \geq 0$ gives us part of the loop invariant.

What remains to prove this that this r_{t+1} computed by the algorithm is in fact equal to $p_{t+1} - q_{t+1}^2$. We prove that as follows.

$$\begin{aligned} RHS &= p_{t+1} - q_{t+1}^2 \text{ (by code)} \\ &= [100p_t + a_{t+1}] - [10q_t + b_{t+1}]^2 \\ &= [100p_t + a_{t+1}] - [100q_t^2 + 20q_t b_{t+1} + b_{t+1}^2] \\ &= [100(p_t - q_t^2) + a_{t+1}] - [10(2q_t) + b_{t+1}] \times b_{t+1} \text{ (by LI}_t\text{)} \\ &= [100(r_t) + a_{t+1}] - [10(2q_t) + b_{t+1}] \times b_{t+1} \\ &= [R_{t+1}] - [s_{t+1}] \\ &= r_{t+1} = LHS. \end{aligned}$$

(b) (8 marks) Determine the running time of algorithms.

- Answer: The number of iterations is denoted n which is the number of pairs of digits when writing down p . This is also $\mathcal{O}(\text{size})$.

Each iteration requires computing and subtracting the strange value $\langle 2q_t, b_{t+1} \rangle \times b_{t+1}$. Because b_{t+1} is only a single digit, multiplying by it takes $\mathcal{O}(n)$ time. Subtracting also takes $\mathcal{O}(n)$ time. This gives a total time of $\mathcal{O}(n^2) = \mathcal{O}(\text{size}^2)$.

As said, I learned this digit by digit algorithm for computing square root in high school. The following is my explanation of how this algorithm looks like long division.

Pre and Post Conditions: Our precondition is that we have an integer p that we want to take the square root of. Our post condition is that we find q which is the square root of p rounded down to the nearest integer, namely the largest integer q such that $q^2 \leq p$ and $r = p - q^2$ is the remainder.

Example: Our working example will be $p = 780147123$, $q = 27931$ and $r = 780147123 - 27931^2 = 6362$. (We can talk later about getting more precision.)

Long Division: The algorithm we will consider is similar to long division.

Like when doing long division, we think of both the input p and the answer q as strings of decimal digits instead of as values. If the square root q has n digits, then $p \approx q^2$ will have twice as many, namely $2n$. For this reason, we start by breaking the digits of our input $p = 780147123$ into pairs as follows, $p = \langle a_1, \dots, a_t, \dots, a_n \rangle = \langle 07, 80, 14, 71, 23 \rangle$. Note that we are using $a_3 = 14 \in [00..99]$ to represent the value of the third pair of digits. The algorithm will consider these pairs a_t from the left to the right. For each such pair, the next digit of the answer $q = 27931 = \langle b_1, \dots, b_t, \dots, b_n \rangle = \langle 2, 7, 9, 3, 1 \rangle$ will be determined. Note that we are using $b_3 = 9$ to represent the third digit of the answer.

$$\begin{array}{r}
 2\ 7\ 9\ 3\ 1 \\
 \sqrt{7\ 80\ 14\ 71\ 23} \\
 \underline{7} \quad \text{2x0 = 0} \\
 -4 \quad \text{02 x 2 = 4} \\
 \underline{3\ 80} \quad \text{2x2 = 4} \\
 -329 \quad \text{47 x 7 = 329} \\
 \underline{51\ 14} \quad \text{2x27 = 54} \\
 -4941 \quad \text{549 x 9 = 4941} \\
 \underline{173\ 71} \quad \text{2x279 = 558} \\
 -16749 \quad \text{5583 x 3 = 16749} \\
 \underline{622\ 23} \quad \text{2x2793 = 5586} \\
 -55861 \quad \text{55861 x 1 = 55861} \\
 \underline{6362}
 \end{array}$$

Loop Invariant: We will use the “more of the input” type of loop invariant. After $t = 2$ iterations, we will have considered the first two pairs of pair of digits of p . If these were the entire input, then we would have the answer. More specifically, our input so far is $p_t = \langle a_1, a_2, \dots, a_t \rangle = \langle 07, 80 \rangle = 0780$, our answer so far is $q_t = \langle b_1, b_2, \dots, b_t \rangle = \langle 2, 7 \rangle = 27$, and our remainder so far is $r_t = 0780 - 27^2 = 51$.

Loop Invariant: The loop invariant maintained is that $p_t = q_t^2 + r_t$.

Step: We are now going to consider the step taken by this algorithm to make progress while maintaining the loop invariant. Progress is that t increases by one, so the next pair of pair of digits $a_{t+1} = 14$ is considered changing our current input from $p_t = \langle a_1, a_2, \dots, a_t \rangle = \langle 07, 80 \rangle = 780$ to $p_{t+1} = \langle a_1, a_2, \dots, a_{t+1} \rangle = \langle 07, 80, 14 \rangle = 78014$. Our goal is then to determine the next digit $b_{t+1} = 9$ in order to change our current answer from $q_t = \langle b_1, b_2, \dots, b_t \rangle = \langle 2, 7 \rangle = 27$ to $q_{t+1} = \langle b_1, b_2, \dots, b_{t+1} \rangle = \langle 2, 7, 9 \rangle = 279$.

Code: $p_{t+1} = 100p_t + a_{t+1}$ and $q_{t+1} = 10q_t + b_{t+1}$.

Extended Remainder R_{t+1} : By the loop invariant, we know the remainder is $r_t = 780 - 27^2 = 51$. Looking at the input p , we can see that the next pair of pair of digits is $a_{t+1} = 14$. As done in long division, we will “bring down” this 14 and concatenate it onto the remainder giving $R_{t+1} = \langle r_t, a_{t+1} \rangle = \langle 51, 14 \rangle = 5114$. See the circle containing a 14 in the above figure.

Code: $R_{t+1} = 100r_t + a_{t+1}$

Strange Value s_{t+1} : As in long division, though we still do not know the next digit $b_{t+1} = 9$, we will use it to compute some strange value $s_{t+1} = \langle 2q_t, b_{t+1} \rangle \times b_{t+1} = 4941$, which is written under the value R_{t+1} . This strange number s_{t+1} is computed on the right of the figure. In this, the digit b_{t+1} underlined. Start by doubling the current answer $q_t = 27$ to get 54. Then concatenate the new (unknown) digit $b_{t+1} = 9$ onto the right of this giving $\langle 2q_t, b_{t+1} \rangle = \langle 54, 9 \rangle = 549$. Then multiply this by b_{t+1} to get $s_{t+1} = 4941$.

Remainder r_{t+1} : Again, like in long division, we subtract to get the new remainder $r_{t+1} = R_{t+1} - s_{t+1} = 5114 - 4941 = 173$.

Code: $s_{t+1} = \langle 2q_t, b_{t+1} \rangle \times b_{t+1} = (10(2q_t) + b_{t+1}) \times b_{t+1}$ and $r_{t+1} = R_{t+1} - s_{t+1}$.

New Digit b_{t+1} : Our goal now is to determine the next digit $b_{t+1} = 9$ of the answer. The digit $b_{t+1} \in [0..9]$ is chosen to be the biggest digit for which this remainder r_{t+1} is not negative.

One Decimal at a Time: We also always want each digit b_{t+1} computed to be in $[0..9]$. We show that 10 would be too big for b_{t+1} by plugging it into the computation for r_{t+1} . Also plug in $a_{t+1} < 100$ and $r_t \leq 2q_t$. This gives $r_{t+1} = 100r_t + a_{t+1} - 20q_t b_{t+1} - b_{t+1}^2 < 100(2q_t) + (100) - 20q_t(10) - (10)^2 = 0$. Given we made b_{t+1} small enough that r_{t+1} would be non-negative, we know that b_{t+1} must be less than 10, i.e. in $[0..9]$.

Binary: If this algorithm was done in binary instead of base 10, then each digit b_{t+1} to find would either be zero or one. This would make the computation easier.

Precision: (Not required for the test.) The algorithm described above returns the square root of p rounded down to the nearest integer with some remainder. Here we bound how big that remainder could be. Note that we could make the loop invariant $p = q^2 + r$ true simply by setting $q = 0$ and $r = p$, but then the remainder would be way too big. Instead, we want q to be the biggest integer such that r is non-negative. This is obtained each step by explicitly making b_{t+1} be the biggest integer so that r_{t+1} is non-negative. This in turn makes q_{t+1} the biggest such integer. This gives us the corollary that if q_t was made one bigger, then the remainder would be negative, i.e. that $(q_t + 1)^2 > p_t$. Expanding this and using the LI that $p_t = q_t^2 + r_t$ gives that $q_t^2 + 2q_t + 1 > q_t^2 + r_t$ or that $r_t < 2q_t + 1$. Because r_t is an integer, we know that $r_t \leq 2q_t$. (Oh I guess we got lucky with $q = 27931$ and r only 6362.) Though bounded, this is still quite large.

More precision can be obtained simply by shifting the decimal. Suppose the input is a real number p and you want to find its square root q with the m digits to the right of the decimal point correct and as it turns out the remainder r being at most $2 \cdot 10^{n-m}$, where as before $2n$ is the number of digits left of the decimal point in p , (i.e. at most 10^{2n}). For example, $p = 780.147123$ and $m = 3$ gives $q = 27.931$ and $r = 780.147123 - 27.931^2 = 0.006362 \leq 2 \cdot 10^{1.5-3} = 0.063$. We can accomplish this as follows. First shift the decimal point of input p by $2m = 6$ places from $p = 780.147123$ to $p' = 780147123$. Then execute the above algorithm giving $q' = 27931$ and $r' = 780147123 - 27931^2 = 6362$. Then shift the decimal place in p' and r' back $2m$ places and that in q' only m places to give $r = 780.147123 - 27.931^2 = 0.006362$. We know that p has at most $2n$ decimal places to the left. Hence p' has at most $2n + 2m$ and its square root q' has at most $n + m$. We bounded $r' \leq 2q'$ and hence (ignoring the 2) it has at most $n + m$ as well. Shifting the decimal in r' to left by $2m$ gives r at most $(n + m) - 2m = n - m$ decimal places to the left and hence bounded by $2 \cdot 10^{n-m}$ as desired.

Reference:

See

- https://en.wikipedia.org/wiki/Methods_of_computing_square_roots#Digit-by-digit_calculation
- <https://www.wikihow.com/Calculate-a-Square-Root-by-Hand>