CSE 3101 Design and Analysis of Algorithms
Practice Test for Unit 1
Loop Invariants and Iterative Algorithms

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First learn the steps. Then try them on your own. If you get stuck only look at a little of the answer and then try to continue on your own.

1. (Answer in slides)
   "And the last shall be first!"
   (For each of A-K see the figure on the next page labeled with the corresponding letter. Here $n = 6$.)

   A: Precondition; The input to this problem is a linked list of n nodes in which the last node points back to the first forming a circle. The variable *last* points at the "last" node in the list. The nodes are of type $Node$ containing a field $info$ and a field *next* of type $Node$ which is to point at the next node in the list. The values $1, 2, \ldots, 6$ in figure are just to help you and cannot be used in the code.

   B: Postcondition; The required output to this problem consists of the same nodes in the same circle, except each node now points at the "previous" node instead of the "next". In other words the pointers are turned from clockwise to counter clockwise. The variable *last* still points at the same node, but this "last" node has become "first".

   C: Loop Invariant$_t$; Your algorithm for this problem will be iterative (i.e. a loop taking one step at a time). Your first task is to give the loop invariant. This is not done in words but by **drawing** what the data structure will look like after the algorithm has executed it's loop $t = 2$ times. Hint: Nodes with value 1 and 2 have their pointers fixed. Hint: You may need a couple of additional pointers to hold things in place. Give them as meaningful names as possible.

   D: Loop Invariant$_{t+1}$; The next task in developing an iterative algorithm is:

   - Maintain the Loop Invariant:
   We arrived at the top of the loop knowing only that the Loop Invariant is true and the Exit Condition is not.
   We must take one step (iteration) (making some kind of progress).
   And then prove that the Loop Invariant will be true when we arrive back at the top of the loop.
   $\langle loop{-}invariant_t \rangle \ \&\ not\, \langle exit{-}cond \rangle \ \&\ code_{loop} \ \Rightarrow\ \langle loop{-}invariant_{t+1} \rangle$

   Towards this task, **draw** what the data structure will look like after the algorithm has executed it's loop one more time, i.e. $t{+}1 = 3$ times.

   E: Code C to D; In space E, give me the code to change the data structure in figure C into that in figure D. This will be the code within the loop that makes progress while maintaining the loop invariant. Assume every variable and field is public.

   F: Loop Invariant$_0$; The next task in developing an iterative algorithm is:

   - Establish the Loop Invariant:
   Our computation has just begun and all we know is that we have an input instance that meets the Pre Condition.
   Being lazy, we want to do the minimum amount of work.
   And to prove that it follows that the Loop Invariant is then true.
   $\langle pre{-}cond \rangle \ \&\ code_{pre\text{-}loop} \ \Rightarrow\ \langle loop{-}invariant \rangle$

   Towards this task, **draw** what the data structure will look like when the algorithm is at the top of the loop but has has executed this loop zero times. Make it as similar as possible to the precondition so that minimal work needs to be done in G.

   G: Code A to F; In space G, give me the code to change the data structure in figure A into that in figure F. This will be the initial code that establishes the loop invariant.

H: Loop Invariant$_n$; The next task in developing an iterative algorithm is:

- Obtain the Post Condition:
We know the Loop Invariant is true because we have maintained it.
We know the Exit Condition is true because we exited.
We do a little extra work.
And then prove that it follows that the Post Condition is true.
$\langle loop-invariant \rangle$ & $\langle exit-cond \rangle$ & $code_{post\text{–}loop} \Rightarrow \langle post-cond \rangle$

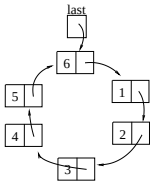
Towards this task, **draw** what the data structure will look like after the algorithm has executed it's loop n times. Make it as similar as possible to the postcondition so that minimal work needs to be done in J.

I: Exit Condition: What is the exit condition, i.e. how does your code recognize that it is in the state you drew in H, given that the algorithm does not know n (and did not count t) and does not know the values $1, 2, \ldots, 6$. Also be careful that your exit condition does not have you drop out at state D.

J: Code H to B; In space J, give me the code to change the data structure in figure H into that in figure B. This will be the final code that establishes the post condition.

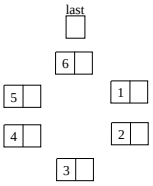
K: Give the complete code; Put all the code together into a routine solving the problem.

A: Precondition

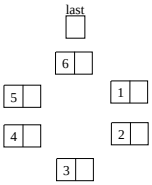


G: Code A to F

F: Loop Invariant $_0$

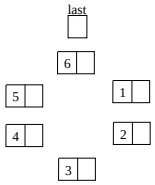


C: Loop Invariant $_t$

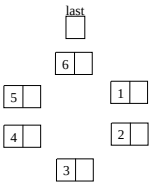


E: Code C to D

D: Loop Invariant $_{t+1}$



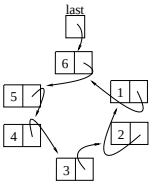
H: Loop Invariant $_n$



I: Exit Condition

J: Code H to B

B: Post Condition



K: Give the complete code

2. (Answer in slides) You are in the middle of a lake of radius one. You can swim at a speed of one and can run infinitely fast. There is a smart monster on the shore who can't go in the water but can run at a speed of four. Your goal is to swim to shore arriving at a spot where the monster is not and then run away. If you swim directly to shore, it will take you 1 time unit. In this time, the monster will run the distance $\Pi < 4$ around to where you land and eat you. Your better strategy is to maintain the most obvious loop invariant while increasing the most obvious measure of progress for as long as possible and then swim for it. Describe how this works.

3. Iterative Cake Cutting: The famous algorithm for fairly cutting a cake in two is for one person to cut the cake in the place that he believes is half and for the other person to choose which "half" he likes. One player may value the icing more while the other the cake more, but it does not matter. The second player is guaranteed to get a piece that he considers to be worth at least a half because he choose between two pieces whose sum worth for him is at least a one. Because the first person cut it in half according to his own criteria, he is happy which ever piece is left for him. Our goal is write an iterative algorithm which solves this same problem for n players.

   To make our life easier, we view a cake not as three dimensional thing, but as the line from zero to one. Different players value different subintervals of the cake differently. To express this, he assigns

some numeric value to each subinterval. For example, if player p_i's name is written on the subinterval $[\frac{i-1}{2n}, \frac{i}{2n}]$ of cake then he might allocate a higher numeric value to it, say $\frac{1}{2}$. The only requirement is that the sum total value of the cake is one.

Your algorithm is only allowed the following two operations. In an evaluation query, $v = Eval(p, [a, b])$, the algorithm asks a player p how much v he values a particular subinterval $[a, b]$ of the whole cake $[0, 1]$. In a cut query, $b = Cut(p, a, v)$, the protocol asks the player p to identify the shortest subinterval $[a, b]$ starting at a given left endpoint a, with a given value v. In the above example, $Eval(p_i, [\frac{i-1}{2n}, \frac{i}{2n}])$ returns $\frac{1}{2}$ and $Cut(p_i, \frac{i-1}{2n}, \frac{1}{2})$ returns $\frac{i}{2n}$. Using these the two player algorithm is as follows.

algorithm $Partition2(\{p_1, p_2\}, [a, b])$

⟨*pre−cond*⟩: p_1 and p_2 are players.
$[a, b] \subseteq [0, 1]$ is a subinterval of the whole cake.

⟨*post−cond*⟩: Returns a partitioning of $[a, b]$ into two disjoint pieces $[a_1, b_1]$ and $[a_2, b_2]$ so that player p_i values $[a_i, b_i]$ at least half as much as he values $[a, b]$.

begin
$v_1 = Eval(p_1, [a, b])$
$c = Cut(p_1, a, \frac{v_1}{2})$
if($Eval(p_2, [a, c]) \leq Eval(p_2, [c, b])$) then
$[a_1, b_1] = [a, c]$ and $[a_2, b_2] = [c, b]$
else
$[a_1, b_1] = [c, b]$ and $[a_2, b_2] = [a, c]$
end if
return($[a_1, b_1]$ and $[a_2, b_2]$)
end algorithm

The problem that you must solve is the following

algorithm $Partition(n, P)$

⟨*pre−cond*⟩: P is a set of n players.
Each player in P values the whole cake $[0, 1]$ by at least one.

⟨*post−cond*⟩: Returns a partitioning of $[0, 1]$ into n disjoint pieces $[a_i, b_i]$ so that for each $i \in P$, the player p_i values $[a_i, b_i]$ by at least $\frac{1}{n}$.

begin
...
end algorithm

(a) Can you cut off n pieces of cake, each of *size* strictly bigger than $\frac{1}{n}$, and have cake left over? Is it sometimes possible to allocated a disjoint piece to each player, each worth by the receiving player much more than $\frac{1}{n}$, and for there to still be cake left? Explain.

(b) Give **all** the required steps to describe this LI algorithm. (Even if you do not know how to do a step for this algorithm, minimally state the step.)

As a big hint to designing an iterative algorithm, we will tell you what the first iteration accomplishes. (Later iterations may do slightly modified things.) Each player specifies where he would cut if he were to cut off the first $\frac{1}{n}$ fraction of the $[a, b]$ cake. The player who wants the smallest amount of this first part of the cake is given this piece of the cake. The code for this is as follows.

loop $i \in P$
$c_i = Cut(p_i, 0, \frac{1}{n})$
end loop
i_{min} = the $i \in P$ that minimizes c_i
$[a_{i_{min}}, b_{i_{min}}] = [0, c_{i_{min}}]$

As a second big hint, your loop invariant should include:

i. How the cake has been cut so far.

ii. Who has been given cake and how do they feel about it.

iii. How do the remaining players feel about the remaining cake.

4. Tiling Chess Board: You are given a $2n$ by $2n$ chess board. You have many tiles each of which can cover two adjacent squares. Your goal is to place non-overlapping tiles on the board to cover each of the $2n \times 2n$ tiles except for to top-left corner and the bottom-right corner. Prove that this is impossible. To do this give a loop invariant that is general enough to work for any algorithm that places tiles. Hint: chess boards color the squares black and white.

5. Loop Invariants - Sorted Matrix Search: Search a Sorted Matrix Problem: The input consists of a real number x and a matrix $A[1..n, 1..m]$ of nm real numbers such that each row $A[i, 1..m]$ is sorted from left to right and each column $A[1..n, j]$ is sorted from top to bottom. The goal is to determine if the key x appears in the matrix. Design and analyze an iterative algorithm for this problem that examines as few matrix entries as possible. Careful, if you believe that a simple binary search solves the problem. Later we will ask for a lower bound and for a recursive algorithm.

6. $d + 1$ Colouring: Given an undirected graph G such that each node has at most d neighbors, colour each node with one of $d + 1$ colours so that for each edge the two nodes have different colours. Hint: Don't think too hard. Just colour the nodes. What loop invariant do you need?
Hint: All it will say is that you have not gone wrong yet.
Give all the steps done in class to develop this iterative algorithm.

7. Loop Invariants - Connected Components: The input is a matrix I of pixels. Each pixel is either black or white. A pixel is considered to be connected to the four pixels left, right, up, and down from it, but not diagonal to it. The algorithm must allocated a unique name to each connected black component. (The name could simply be 1,2,3,..., to the number of components.) The output consists of another matrix $Names$ where $Names(x, y) = 0$ if the pixel $I(x, y)$ is white and $Names(x, y) = i$ if the pixel $I(x, y)$ is a part of the component named i. The algorithm reads the matrix from a tape row by row as follows.

```
loop y = 1...h
    loop x = 1...w
        ⟨loop−invariant⟩: ??
        if( I(x,y)=white )
            Names(x,y) = 0
        else
            ???
        end if
    end loop
end loop
end algorithm
```

The image may contain spirals and funny shapes. Connected components may contain holes that contain other connected components. A particularly interesting case is the image of a comb consisting of many teeth held together at the bottom by a handle. Scanning the image row by row, one would would first see each tooth as a separate component. As the handle is read, these teeth would merge into one.

(a) Give the classic *more of the input loop invariant* algorithm. Don't worry about its running time.

(b) This version of the question is easier in that the matrix $Names$ need not be produced. The output is simply the number of connected black components in the image. However, this version of the question is harder in your computation is limited in the amount of memory it can use. For

example, you don't remember pixels that you have read if you do not store them in this limited memory and you don't have nearly enough memory to store them all. The number of components may be Θ(the pixels) so you cant store all of them either. How little memory can you get away with?

(c) In this this final version, in addition to a small amount of fast memory, you have a small number of tapes for storing data. Data access on tapes, however, is quite limited. A tape is in one of three modes and cannot switch modes mid operation. In the first mode, the data on a tape is read forwards one data item at a time in order. The second mode, is the same except it is read backwards. In the third mode, the tape is written to. However, the command is simply $write(data)$ which appends this data to the end of the tape. Data on a tape cannot be changed. All you can do is to erase the entire tape and start writing again from the beginning. An algorithm must consist of a number of passes. The first pass reads the input from the input tape one pixel at a time row by row in order. As it goes, the algorithm updates what is store in fast memory and outputs data in order onto a small number output tapes. Successive passes can read what was written during a previous pass and/or the input again. The last pass must write the required output $Names$ onto a tape. You want to use as little fast memory and as few passes as possible. For each pass clearly state the loop invariant.

8. A *tournament* is a directed graph (see Section **??**) formed by taking the complete undirected graph and assigning arbitrary directions on the edges, i.e., a graph $G = (V, E)$ such that for each $u, v \in V$, exactly one of $\langle u, v\rangle$ or $\langle v, u\rangle$ is in E. A *Hamiltonian path* is a path through a graph that can start and finish any where but must visit every node exactly once each. Design an algorithm which finds a Hamiltonian path through it given any tournament. Because this algorithm finds a Hamiltonian path for each tournament, this algorithm, in itself, acts as proof that every tournament has a Hamiltonian path.

9. An *Euler tour* in an undirected graph is a cycle that passes through each edge exactly once. A graph contains an Eulerian cycle iff it is connected and the degree of each vertex is even. Given such a graph find such a cycle.

10. (Answer in slides) The ancient Egyptians and the Ethiopians had advanced mathematics. Merely by halving and doubling, they could multiply any two numbers correctly. Say they want to buy 15 sheep at 13 Ethiopian dollars each. Here is how he figures out the product. Put 13 in a left column, 15 on the right. Halve the left value; you get $6\frac{1}{2}$. Ignore the $\frac{1}{2}$. Double the right value. Repeat this (keeping all intermediate values) until the left value is 1. What you have is

13	15
6	30
3	60
1	120

Even numbers in the left column are evil and, according the story, must be destroyed, along with their guilty partners. So scratch out the 6 and its partner 30. Now add the right column giving $15 + 60 + 120 = 195$, which is the correct answer. Give all the required steps to describe this LI algorithm. The following are some hints.

(a) Write pseudo code, which given two positive integers x and y follows this procedure and outputs the resulting value. Part of the loop invariant is that the variable ℓ holds the current left value, r the current right, and s the sum of all previous right values that will be included in the final answer.

Break the algorithm within the loop into two steps. In the first step, if ℓ is odd it decreases by one. In the second step, ℓ (now even) is divided by two. These steps, must update r and s as needed.

(b) Give a meaningful loop invariant relating the current values of ℓ, r, s, x, and y. (Hint, look at the Shrinking Instance loop invariant.) In addition to this invariant being true every time the computation is at the top of the loop, it will also be true every time the computation is between the first and the second steps of each iteration.

(c) Draw pictures to give a geometric explanation for the steps.

(d) What is the Ethiopian exit condition? How might you improve on this? How does the exit condition, the loop invariant, and perhaps some extra code, establish the post condition?

(e) Suppose that the input instance x and y are each n bit numbers. How many bit operations are used by your algorithm as a function of n? (Adding two n' bit numbers requires $\mathcal{O}(n')$ time.) Suppose the Ethiopians counted with pebbles. How many pebble operations does their algorithm require? How do these times compare? How do these times compare with the high school algorithm for multiplying? How do they compare to laying out a rectangle of x by y pebbles and them counting them?

(f) This algorithm seems very strange. Compare it to using the high school algorithm for multiplying in binary.

11. (Answer in slides) Multiplying using Adding: My son and I wrote a JAVA compiler for his grade 10 project. We needed the following. Suppose you want to multiply $X \times Y$ two n bit positive integers X and Y. The challenge is that the only operations you have are adding, subtracting, and tests like $\leq$. The standard high school algorithm seems to require looking at the bits of X and Y and hence can't obviously be implemented here. The Ethiopian algorithm requires dividing by two, so can't be implemented either. A few years after developing this algorithm, I noticed that it is in fact identical to this high school multiplication algorithm, but I don't want to tell you this because I don't want you thinking about the algorithm as a whole. This will only frighten you. All you need to do is to take it one step at a time. I want only want you to establish and maintain the loop invariant and use it to get the post condition. (The flavor is very similar to what was done in the Ethiopian problem on the practice test.) To help, I will give you the loop invariants, the measure of progress, and the exit condition. I also want you to explain what the time complexity of this algorithm is, i.e. the number of iterations and the total number of bit operations as a function of the size of the input.

We have values i, x, a, $u[]$, and $v[]$ such that

Useful Arrays: Before the main loop, I will set up two arrays for you with the following values. Then they will not be changed.

LI0': $u[j] = 2^j$, (for all j until $u[j] > X$)

LI0": $v[j] = u[j] \times Y$. Note $v[j] - u[j] \times Y = 0$

My Code: For completion, I include my code, but it is not necessary for you to understand it.

$u[0] = 1$
$v[0] = Y$
$j = 0$
while($u[j] \leq X$)
    $u[j+1] = u[j] + u[j]$
    $v[j+1] = v[j] + v[j]$
    $j = j + 1$
end while

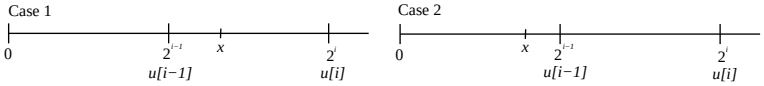
For Main Loop: These are the loop invariants for you to deal with.

LI0: Neither X nor Y change.

LI1: $X \times Y = x \times Y + a$ (i.e. Shrinking Instance)

LI2: $x \geq 0$

LI3: $x < 2^i = u[i]$ (below are two cases)



Measure of progress: i. It decreases by 1 each iteration.

Exit Condition: $i = 0$

Use the steps laid out in class to complete the description of the algorithm.