York University
CSE 2001 – Unit 5.0 Uncountable
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Don't cheat by looking at these answers prematurely.

1. What is the "size" of the following sets:

i	$\{i \mid i$ is a prime integer$\}$:	Finite	Countably-Infinite	Uncountable
ii	$\{G \mid G$ is a grammar$\}$:	Finite	Countably-Infinite	Uncountable
iii	$\{L \mid L$ is a regular language$\}$:	Finite	Countably-Infinite	Uncountable
iv	$\{L \mid L$ is a language$\}$:	Finite	Countably-Infinite	Uncountable
v	all points on a 1 inch line:	Finite	Countably-Infinite	Uncountable
vi	all the atoms in the earth:	Finite	Countably-Infinite	Uncountable

   - Answer: i, ii, and iii: each object in each have an easy finite description and hence the sets are countable. iv and v: each object in each need an infinite description and hence the sets are uncountable. vi: finite

2. Write pseudo code that loops over every tuple of 4 positive integers $\langle w, x, y, z\rangle$ such that each is eventually printed (and only printed once).

   - Answer:

     algorithm $Print4Tuples()$

     $\langle$***pre−cond***$\rangle$**:** No inputs

     $\langle$***post−cond***$\rangle$**:** Each tuple $\langle w, x, y, z\rangle$ is eventually printed.

```
begin
    loop sum ≥ 0
        w = 0 ... sum
            x = 0 ... sum − w
                y = 0 ... sum − w − x
                    z = sum − w − x − y
                    Print(⟨w, x, y, z⟩)
end algorithm
```

3. Prove that there are more real numbers than integers, i.e. $|\mathcal{R}| > |\mathcal{N}|$.

   - Answer: We prove this by proving the following first order logic statement
     $\forall$ an inverse functions F^{-1} from $\mathcal{N}$ ideally to $\mathcal{R}$,
     $\exists x_{diagonal} \in \mathcal{R}, \forall i \in \mathcal{N}, F^{-1}(i) \neq x_{diagonal}$
     namely there are not enough integers to hit each real.
     We prove this by playing the game.
     Let F^{-1} be an arbitrary inverse function from $\mathcal{N}$ ideally to $\mathcal{R}$.
     Define the real $x_{diagonal} \in \mathcal{R}$ as follows.
     For each $i \in \mathcal{N}$, I must define the i^{th} digit of $x_{diagonal}$.
     For this, we use flip of the i^{th} diagonal element as follows.
     Let x_i denote the real $F^{-1}(i)$ that the i^{th} row gives us.
     Let d_i denote the i^{th} digit of x_i.
     Then let the i^{th} digit of $x_{diagonal}$ be any digit d'_i other than d_i.
     This completely defines $x_{diagonal}$.
     Continuing the game, let $i \in \mathcal{N}$ be arbitrary.
     Note $x_i = F^{-1}(i)$ and $x_{diagonal}$ differ in their i^{th} digits.
     This proves that $F^{-1}(i) \neq x_{diagonal}$.

4. Rationals:

   (a) Each fraction has an infinite description, eg $\frac{1}{3} = 0.33333\ldots$. Didn't we say that this means the set of fractions $\mathcal{Q}$ is uncountable? Explain why or why not.

      - Answer: What we said is that if each element in a set S can be given a unique finite description then the set S is countable. This does not mean that there are not also infinite descriptions for these elements. The converse is that if S is uncountable that there is no unique finite description, because each element has "contains" an infinite amount of information.

   (b) Study the prove that the set of real numbers is uncountable. Use the exact same proof to show that $\mathcal{Q}$ is uncountable. What if anything goes wrong in the proof?

      - Answer: The argument works the same for $\mathcal{Q}$ until the punchline. The new number constructed is not necessarily rational, so there is no contradiction from the fact that it is missing.

5. Power Sets: Let U be a set of objects. In this question we will first have it be the set of positive integers and then be the set of positive reals less than one. The power set of U is the set of all subsets of U. If U is finite, then its power set has cardinality $2^{|U|}$ elements. Hence, power set of U is often denoted by $2^U = \{s \mid s \subseteq U\}$. Similarly, denote $2^U_{finite} = \{s \mid s \subseteq U \text{ and } |s| \text{ is finite}\}$. We will consider the relative sizes of U, 2^U_{finite}, and 2^U.

   (a) Recall that we say $|2^U_{finite}| \leq |U|$ if $\exists$ a function $F : 2^U_{finite} \rightarrow U$ such that $\forall s \in 2^U_{finite}, F(s) \in U$ and $\forall s, s' \in 2^U_{finite},\ s \neq s' \Rightarrow F(s) \neq F(s')$. A common way to prove $s \neq s' \Rightarrow F(s) \neq F(s')$ is to provided the inverse function F^{-1} and prove that $\forall s\ F^{-1}(F(s) = s$. (Though I did want you think about this, I don't ask you to do it.)

      i. Let $U = \mathcal{N}$ denote the set of positive integers. Then $2^{\mathcal{N}}_{finite}$ denotes the set of finite subsets of $\mathcal{N}$. Prove that $2^{\mathcal{N}}_{finite}$ is countable by defining a concrete function $F(s) = u_s$ mapping each finite subsets s of the positive integers to a unique integer u_s. Use the ascii technique given in the slides.
      - If $s = \{24, 8\}$, what is $F(s)$?
      - What happens if you don't encode the commas?
      - Does the fact that the integers in s can be put into different orders create a problem?
      - What two properties of s are key in proving that for every $s \in 2^{\mathcal{N}}_{finite}$, $F(s)$ is a finite integer?
      - Look at the ASCII table. Why might I have gotten nervous about using the decimal code instead of the hex code?

         - Answer: - Given a set $s \in 2^{\mathcal{N}}_{finite}$ like $s = \{24, 8\}$, map this to the ascii string "$\{24, 8\}$" by ordering the elements of the set s in an arbitrary order, writing each such integer in decimal notation, separating them with commas, and just for aesthetics, surrounding it all with open and close curly bracket characters. It should be clear that this provides each such object s with (at least one) finite description that such that each such description uniquely identifies s. From here the technique is exactly like in the slides. Each character in the string description "$\{24, 8\}$" is converted into its hex ascii. Each hex ascii is two digits. Concatenate these digits into one string. Then view this as the single hex integer $F(s) = u_s$. For example, $F(\{24, 8\}) = 7B32342C387D_{16}$.
         - If you don't encode the commas, than the description string "$\{24, 8\}$" becomes only "248" which could have been produced from $F(\{2, 4, 8\})$ and $F(\{2, 48\})$ as well as $F(\{24, 8\})$.
         - The fact that the integers in s can be put into different orders does not create a problem because it does not matter if one object s gets mapped to more than one description, namely "$\{24, 8\}$" vs "$\{8, 24\}$", and hence to more than on integer, namely $F(\{24, 8\}) = 7B32342C387D_{16}$ vs $7B382C32347D_{16}$.

- Each integer in s is represented by a finite number of decimal digits and s contains a finite number of integers, hence the encoding contains a finite number of hex digits and hence $F(s)$ is valid integer.
- Each hex ascii code has two digits. Hence, (when defining F^{-1}), the integer $7B32342C387D_{16}$ can easily be broken into ascii codes words $7B$ 32 34 $2C$ 38 $7D$. In contrast, some decimal ascii codes have 1, others 2 and yet others 3. Hence the string "ww" gets converted to 119 119 and concatenated to 119119. But this could be interpreted as 11 91 19 which is the string "VT [DC3". Ok maybe there is not really a danger of this misunderstanding.

A Previous Answer: Let the binary expansion of $F(s) = u_s$ have a 1 in the i^{th} digit if an only if the positive integer i is in the set s. For every $s \in 2^{\mathcal{N}}_{finite}$, s is finite and hence contains a maximum finite integer u_{max}. It follows that $F(s) \leq 2^{u_{max}+1}$. If s and $s' \in 2^{\mathcal{N}}_{finite}$ are different then there is some positive integer i that is in one but not the other. Hence, $F(s)$ and $F(s')$ differ in that bit and hence must be different integers.

ii. Let $\mathcal{R}_{[0,1)}$ denote the set of positive reals less than one. Let $2^{\mathcal{R}_{[0,1)}}_{finite}$ denote the set of finite subsets of $\mathcal{R}_{[0,1)}$. Let $\mathcal{R}$ denote the set of reals (possibly bigger than one). Prove that $2^{\mathcal{R}_{[0,1)}}_{finite}$ has cardinality at most that of $\mathcal{R}$ by defining a concrete function $F(s) = x_s$ mapping each finite subsets s of the positive reals less than one to a unique real x_s. Be as explicit as you can. Hint: Interweave the bits. Be sure (but don't prove) that for your construction, for every $s \in 2^{\mathcal{N}}_{finite}$, $F(s)$ is valid real number and that if s and $s' \in 2^{\mathcal{R}}_{finite}$ are different then $F(s) \neq F(s')$.

- Answer: Let s be a finite subset of positive reals less than one. Because $F(s) = x_s$ can be bigger than one and because $n = |s|$ is a finite integer, let the integer part of x_s be $|s|$. Let the i^{th} bit of the j^{th} largest real in s specify the $k = i \times |s| + j^{th}$ bit of x_s. For example, if $s = \{0.111..., 0.222..., 0.333...\}$. then $x_s = 3.123123123....$

(b) A Hierarchy of infinities.

i. Prove that for every set U, the cardinality of 2^U is strictly bigger than that of U, i.e. $|2^U| > |U|$. In our third definition of $|2^U| \leq |U|$, we argued that if each object $u \in U$ is able to hit at most one element $F^{-1}(u) = s \in 2^U$ and this process manages to hit every element $s \in 2^U$, then it follows that $|2^U| \leq |U|$. Conversely, we prove $|2^U| > |U|$, by proving that $\forall$ inverse functions F^{-1} from U ideally to 2^U, $\exists s_{new} \in 2^U, \forall u \in U, F^{-1}(u) \neq s_{new}$.
Your proof should use the first order logic game between the adversary and the prover. Note unlike the proof that $|R| > |N|$, the set U might not be countable and hence can't be listed and hence the diagonal can be visualized.
Hint: Woody Allen once said that he did not want to be a member of any club that would have him as a member. In this spirit, for each person, put him in the heaven club iff he is not in the club that is mapped to him.

- Answer: Let $F^{-1}(u) = s_u$ be an arbitrary mapping from an object $u \in U$ ideally to a subset s_u of U.
We construct as follows a set $s_{new} \in 2^U$. For each $u \in U$, there are two cases. If $F^{-1}(u) = s_u \in 2^U$, then put $u \in s_{new}$ if an only if u is not in the set s_u. On the other hand, if $F^{-1}(u) \notin 2^U$, then it does not matter if you put u in s_{new} or not.
Let u be an arbitrary object in s.
We prove $F^{-1}(u) \neq s_{new}$ as follows. There are two cases. If $F^{-1}(u) = s_u$, then s_{new} and s_u differ in whether or not they contain u. Hence, $s_{new} \neq s_u = F^{-1}(u)$. On the other hand, $F^{-1}(u) \notin 2^U$, then clearly it is not equal to $s_{new} \in 2^U$.

ii. We can use the previous theorem that for every set U, $|2^U| > |U|$ to get many great results. For example, if U is the set of natural numbers $\mathcal{N}$, then we get that $|2^{\mathcal{N}}| > |\mathcal{N}|$ giving that the set $2^{\mathcal{N}}$ of subsets of $\mathcal{N}$ is uncountable. (We did not prove it, but $2^{\mathcal{N}}$ has the same cardinality as the reals $\mathcal{R}$.) As a second example, let U be the set of reals $\mathcal{R}$, then we get that $|2^{\mathcal{R}}| > |\mathcal{R}|$ giving that the set $2^{\mathcal{R}}$ of subsets of $\mathcal{R}$ is a bigger infinity than the number of reals. Prove by that there are a hierarchy of an infinite number of different sizes of infinity.

- Answer: For each size of infinity, let U be a set of this cardinality. The theorem then gives that 2^U has an even bigger cardinality.

6. Computable and Describable Reals

(a) A real number x is said to be *computable* if there is a Java program that on input zero prints out the decimal representation of x from left to right. Note that at no point in time will all of the digits of x be printed out, but for each digit of x, there will be an eventual time at which this digit will be printed out. Use the words taught in class to prove in one sentence whether or not all real numbers computable.

- Answer: There are countably many Java program and uncountably many reals. Hence, most reals are not computable.

(b) A real number x is said to be *describable* if it can be unambiguously denoted by a finite piece of English text. For example, $x = 2$ is described as "Two" and $x = \Pi$ as "The area of a circle of radius one." Use the words taught in class to prove in one sentence whether or not all real numbers describable.

- Answer: There are countably many finite pieces of English text and uncountably many reals. Hence, most reals are not describable.

(c) Prove that every computable real is also describable.

- Answer: Let x be a computable real that is output by a program P. The following is an unambiguous denotation: "The real number output by P".

(d) Prove whether or not there a real number that can be described, but not computed? "Let x be the smallest real number that is not computable" is not a valid answer for the same real that "Let x be the smallest real number that is bigger than zero" is ill defined. Hint: Use a diagonalization proof.

- Answer: Consider the following description. "Let M_i be the i^{th} Java description. For each i, let c_i be the i^{th} character that M_i prints if it prints this many characters and zero otherwise. Let x be the real number whose i^{th} binary bit after the decimal is 0 iff $c_i = 1$." Clearly x is describable because I just described it uniquely. It is not, however, computable because for each i, the i^{th} Java program has a different i^{th} character than x's i^{th} bit.
  We did not ask this, but it seems the same proof should prove that you can describe a real that is not describable. Create a list of all describable reals based on the lexicographical order of their descriptions, and then use diagonalization to find a new real not on the list. If this new real is describable due to the previous sentence, then we seem to have a contradiction. I guess there is a hierarchy of "descriptions". You use a higher order description to describe a real not describable by a lower order description.