EECS 1090 – Test 2

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1. Write in English the negation of each of these statements. Move the negation all the way. in.
   Hint: Do not trust your intuition. In your mind, translated the English provided into math, negate the math, and translate the resulting math back into English.

   (a) (5 marks) If it snows today, then I will go skiing tomorrow and have fun.

   - Answer: It snows today, but I will either not go skiing tomorrow or will not have fun.

   (b) (5 marks) There is at least one person who understands every topic in mathematics.

   - Answer: For every person, there is some topic in mathematics that he/she does not understand.

   (c) (5 marks) In every mathematics class there is some student who falls asleep during lectures.

   - Answer: There is a mathematics class during which every student stays awake.

   (d) (deleted) Some students in this class do not like discrete mathematics.

   - Answer: All students in this class like discrete mathematics.

2. (10 marks) Express the statement "There is a student in this class that has taken some course in every department in the school of mathematical sciences" using quantifiers.
   Hint: The answer for "All students in this class love logic" would be
   $\forall s \in \text{StudentsInThisClass LoveLogic}(s)$.

   - Answer: $\exists s \in \text{StudentsInThisClass } \forall d \in \text{DepartmentInTheSchoolOfMathematical}$
     $\exists c \in \text{CoursesIn}(d) \text{ HasTaken}(s, c)$

3. (20 marks) Use the purple table to prove $[(K \to (M\&N)) \ \& \ \neg N] \to \neg K$.
   Use deduction. Do NOT convert the $\to$ into *and* or *or*.

   - Answer:

1)	Deduction Goal: $[(K \to (M\&N)) \ \& \ \neg N] \to \neg K$.	
2)	$(K \to (M\&N)) \ \& \ \neg N$	Assumption/Premise
3)	$K \to (M\&N)$	Separating And 2
4)	$\neg N$	Separating And 2
5)	$\neg(M\&N) \to \neg K$	Contrapositive 3
6)	$\neg(M\&N)$	Building And 4
7)	$\neg K$	Modus Ponens 5 & 6
8)	$[(K \to (M\&N)) \ \& \ \neg N] \to \neg K$.	Conclude deduction.

4. (20 marks) Use the oracle−prover−adversary game to prove the following is *valid*
   $\forall$ Models eg objects and relations
   $$[\exists x \ (G(x)\&A(x)) \ \& \ \forall y \ (C(y) \to \neg G(y))] \to [\exists z(A(z)\&\neg C(z))]$$
   Hint: Follow the proof game from the slides.
   For each of Jeff's speaking bubbles have a line in your proof here.
   Hint: For each of your lines give its line number, who is talking, what they are proving, assuring, providing, or constructing, and an explanation where it comes from referring to previous line numbers.
   Eg. 3: Oracle 2 assures $\forall y \ (C(y) \to \neg G(y))$.

   - Answer:
     1 Adversary gives us an arbitrary Universe.
     2 Oracle 1 assures $\exists x \ (G(x)\&A(x))$.
     3 Oracle 2 assures $\forall y \ (C(y) \to \neg G(y))$
     4 Prover must prove $\exists z(A(z)\&\neg C(z))$

5 Let $x_\exists$ denote to object given by oracle 1.
6 $G(x_\exists)$ (by 2,5, and separating and)
7 $A(x_\exists)$ (by 2,5, and separating and)
8 Oracle 2 states that she assures for all y and hence assures for this $x_\exists$.
8 $C(x_\exists) \rightarrow \neg G(x_\exists)$ (by 3 and 6)
9 $G(x_\exists) \rightarrow \neg C(x_\exists)$ (Contra Positive of 8)
10 $\neg C(x_\exists)$ (Modus Ponens 6 & 10)
11 Prover (3) constructs z to be $x_\exists$ and knows $\neg C(x_\exists)$ from 11.
12 Prover knows $(A(x_\exists)\&\neg C(x_\exists))$ (from 7 & 11)
13 Prover concludes $\exists z(A(z)\&\neg C(z))$

5. (10 marks) Prove the following is *NOT valid*

$$[\exists z(A(z)\&\neg C(z))] \rightarrow [\exists x\ (G(x)\&A(x))\ \&\ \forall y\ (C(y) \rightarrow \neg G(y))]$$

- Answer: We construct a universe in which the statement is false.
  We need the LHS true and the RHS false.
  Let $A(0)$ be true and $C(0)$ false so that the LHS $[\exists z(A(z)\&\neg C(z))]$ is true.
  Let $C(1)$ be true and $G(1)$ true so that $(C(1) \rightarrow \neg G(1))$ is false
  and hence $\forall y\ (C(y) \rightarrow \neg G(y))]$ is false
  and hence $[\exists x\ (G(x)\&A(x))\ \&\ \forall y\ (C(y) \rightarrow \neg G(y))]$ is false
  and hence the entire $\rightarrow$ is false.

6. Easy with a twist.

(a) (6 marks) Lets assume that every universe contains at least one object.
Use the oracle−prover−adversary game to prove is *valid*

$$[\forall y\ P(y)] \rightarrow [\exists x\ P(x)].$$

Hint: One line is 2) Oracle 0 assures us that our universe contains at least one object.

- Answer:
  1 Adversary gives us an arbitrary Universe.
  2 Oracle 0 assures us that our universe contains at least one object.
  3 Oracle 1 assures $\forall y\ P(y)$.
  4 Prover must prove $\exists x\ P(x)$
  5 Oracle 0 constructs $x_\exists$ be any object in the universe.
  6 The prover (2) constructs his object x to be the oracle's $x_\exists$.
  7 Oracle (1) states that she assures for all y and hence assures for this $x_\exists$.
  8 $P(x_\exists)$ (by 3 and 7)
  9 Prover concludes $\exists x\ P(x)$

(b) (9 marks) Here is a puzzle. Suppose the universe contains no objects at all. For each of the following say whether or not it is true and why?

i. $\forall y\ P(y)$.

- Answer: True. We taught that "all elephants in the room are pink" is true because there are no elephants to make it false.

ii. $\exists x\ P(x)$

- Answer: False. There does not exist any objects x.

iii. $[\forall y\ P(y)] \rightarrow [\exists x\ P(x)]$.

- Answer: False. LHS is true and RHS is false.

7. (10 marks) Suppose that $[\forall n_0,\ \exists n > n_0,\ P(n)]$ is true over the positive integers, i.e., an oracle assures us of it.
Might there be exactly ten values n for which the property $P(n)$ is true?
If not how many might there be?
Explain. Give an example of P, eg $P(0)$ is true, $P(1)$ is false, ...

- Answer: There cannot be exactly ten values n for which $P(n)$ is true, because then there would be a maximum such value, call it n_0. The statement then says that there is a bigger value n for which $P(n)$ is true. This is a contradiction. The number of such values is infinite, because for each there is a bigger. If $P(27)$ is true, then there is some value say 82 for which $P(82)$. Then there is another value say 153 for which $P(153)$. And so on for ever.