$\mathrm{EECS} \ 1090 - \mathrm{Test} \ 2 \ (2024)$ Instructor: Jeff Edmonds

You know what to do. For each of the following statements, state whether or not it is valid/tautology and then either prove it or find a universe in which the statement is false. I recommend your proof being a prover/adversary/oracle proof as done in class, but if you really want you can give a formal proof. Even for a formal proof, you might want to add a few lines of explanation about what is happening. Either way you really need to follows the steps lined out in in class. Make your answer as easy to mark as possible, i.e. it should be clear how each line follows from previous lines. A correct but hard to read proof will not get full marks.

*** PLEASE. FOR ME. JUST GO THOUGH THE STEPS OF THE GAME! ***

1. (30 marks) $[\forall x \exists y (A(x) \land B(y))] \rightarrow [\exists y \forall x (A(x) \land B(y))].$

Hints for the game proof:

- This is like the strange one we did in class except for And instead of Or.
- Recall, the issue is whether or not the value of y that exists depends on the value of x given.
- I give a banana and then I give a bad thing.
- Answer: Valid, i.e. true in every model.

The key thing that makes this surprisingly true is that unlike some statement $\alpha(x, y)$, the statement $A(x) \wedge B(y)$ is "separable" into an x and a y part. Which y works is independent of which x is considered. Both expressions are equivalent to $(\forall x \ A(x)) \wedge (\exists y \ B(y))$.

The informal proof is as follows.

0) Assume a worst case model given by an adversary, i.e. universe of objects and predicates A and B.

1) Goal: Prove $[\forall x \exists y (A(x) \land B(y))] \rightarrow [\exists y \forall x (A(x) \land B(y))].$

- 2) By deduction, the oracle assures us of $\forall x \exists y \ (A(x) \land B(y))$.
- 3) Goal: Prove $\exists y \ \forall x \ (A(x) \land B(y))$

4) In order to prove $\exists y$, I need to construct a value y_{\exists} but I don't know how. Hence, I go straight to my oracle (2).

5) The oracle (2) assures me $\forall x$, so I can give her a value, but I don't have a value to give her. Hence, I give her x = banana.

6) Given x = banana, oracle (2) assuring me that $\exists y$, gives me some value y_{\exists} and she assures us that $A(banana) \land B(y_{\exists})$

7) Separating And (6) gives $B(y_{\exists y})$

8) To prove $\exists y \ \forall x \ (A(x) \land B(y))$, I set y to y_{\exists} for which $B(y_{\exists})$ is true.

9) To prove $\forall x \ (A(x) \land B(y_{\exists}))$, let x_{\forall} be an arbitrary value provided by an adversary.

10) My game is completed if I can prove $A(x_{\forall}) \wedge B(y_{\exists})$.

11) In order to prove $A(x_{\forall})$, I need to use oracle (2) a second time. I give her $x = x_{\forall}$.

12) Though I don't actually need it, she gives me some value y'_{\exists} and assures us that $A(x_{\forall}) \land B(y'_{\exists})$. Note we can't use y'_{\exists} in our proof because I promised our adversary that we would use y_{\exists} .

13) Separating And (12) gives $A(x_{\forall})$ which is the part we needed.

- 14) Building And (7,14) gives $A(x_{\forall}) \wedge B(y_{\exists})$
- 15) By Deduction (2-14), we get goal (1) as needed.

• Answer: Here is the formal proof mirroring the game.

1) D	Peduction Goal: $LHS \rightarrow RHS$	
2)	$\forall x \; \exists y \; (A(x) \land B(y))$	Assumption/Premise
3)	$\exists y \ (A(banana) \land B(y))$	Not knowing what to give, we remove \forall with banana
4)	$A(banana) \wedge B(y_{\exists}(banana))$	Remove \exists (Depends on <i>banana</i>)
5)	$B(y_{\exists}(banana))$	Separating And
6)	$\exists y \ (A(x_{adversary}) \land B(y))$	Having $x_{adversary}$ after constructing $y_{\exists}(banana)$,
		we remove \forall (2) with $x_{adversary}$
7)	$A(x_{adversary}) \land B(y_\exists (x_{adversary}))$	Remove \exists (Depends on $x_{adversary}$).
8)	$A(x_{adversary})$	Separating And
9)	$A(x_{adversary}) \land B(y_\exists(banana))$	Build And $(5,8)$

10)	$\forall x \ (A(x) \land B(y_\exists(banana)))$	$\mathrm{Add} \; \forall$
11)	$\exists y \ \forall x \ (A(x) \land B(y))$	Add \exists
12) L	$HS \rightarrow RHS$	Conclude deduction.

• Answer: Here is the kluged formal proof like the type set one given in slides.

1) De	duction Goal: $LHS \rightarrow RHS$				
2)	$\forall x \exists y \ (A(x) \land B(y))$	Assumption/Premise			
	We know that adversary eventually is going to give us an $x_{adversary}$.				
	Being a free variable, let's just use the notation x .				
3)	$\exists y \ (A(x) \land B(y))$	Remove \forall with the free variable x			
4)	$A(x) \wedge B(y_\exists(x))$	Remove \exists (Depends on x)			
5)	A(x)	Separating And (This is our adversary's value)			
6)	$B(y_\exists(x))$	Separating And			
We need to change the $y_{\exists}(x)$ into y_{\exists} in order make it clear					
that which y works is independent of which x is considered.					
	Most marks will go to the lines $7 \& 8$				
7)	$\exists y \; B(y)$	Add \exists because it is true for some term.			
		Possible because term does not depend on x bounded with $\forall x$.			
8)	$B(y_{\exists})$	Remove \exists . Note now there is no free variable x			
9)	$A(x) \wedge B(y_{\exists})$	Build And (5,8)			
10)	$\forall x \ (A(x) \land B(y_{\exists}))$	$\mathrm{Add} \; \forall$			
11)	$\exists y \ \forall x \ (A(x) \land B(y))$	Add ∃			
$12) LHS \to RHS \tag{C}$		Conclude deduction.			

2. (30 marks)

If $(\alpha \lor \beta)$ is valid/tautology, then so is α or β .

 $[\forall x \ (\alpha(x) \lor \beta(x))] \to [(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))]$ Hint: Being an Or, there is no consistency.

One day she may give you this and the next day that.

- Answer: Like when I ask ChatGPT to prove things, people's biggest struggle with the material seems to not be the PROVING but determining whether the statement is true or not. I thought my hints were clear, but I guess not. We have spent a lot of time looking at what a good proof looks like. We have spent less time understanding what a bad proof looks like. I will add some of that here.
- Answer: Not Valid, i.e. false in some model.
 - 1) Construct a counter example model by having only two objects and setting $\alpha(0)$ and $\beta(1)$ both to be true and $\alpha(1)$ and $\beta(0)$ both to be false.
 - 2) By Building Or, $(\alpha(0) \lor \beta(0))$ and $(\alpha(1) \lor \beta(1))$ are both true, but on different predicates.
 - 3) Because there only two objects, we get that $\forall x \ (\alpha(x) \lor \beta(x))$ is true.
 - 4) Because $\alpha(1)$ is false, $(\forall y \ \alpha(y))$ is false
 - 5) Because $\beta(0)$ is false, $(\forall z \ \beta(z))$ is false
 - 6) By Building Or $[(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))]$ is false.
 - 7) By (3) the LHS is true and by (6) the RHS is false. Hence, LHS \rightarrow RHS is false.
 - 8) Hence, the statement is not
- Answer: Jeff's hint was "Hint: Being an Or, there is no consistency. One day she may give you this and the next day that."

"She" is the oracle. This speaks of why she might not be useful. Jeff was trying to hint that this is not a tautology.

It is not.

But let's suppose we think it is an try to prove it using the game.

You should get partial marks based on how much you do get right. Let me try.

It is not going to go well.

We assume and let an oracle assure us of $\forall x \ (\alpha(x) \lor \beta(x))$.

We then must prove $(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))$.

Build the parse tree for this. The root/last/top operation is not \forall . It is the OR!.

In other words, write this as $OR[(\forall y \ \alpha(y)), (\forall z \ \beta(z))]$ and read the expression left to right.

We have to prove one of these statement, but we don't know which. (This causes us a little worry).

The oracle is assuring us of $\forall x$. So we need to give her a value x. We normally get such values from the adversary.

The game says that we can ask the adversary for an arbitrary value when the root/last/top operation is \forall which it is not. Asking when there is the OR is not really fair to the adversary. He does not know if he is trying to give you a y for which $\alpha(y)$ is not true or a z for which $\beta(z)$ is not true.

Being a nice guy he gives you some arbitrary value x despite it not being his job.

You give the value x to the oracle, who assures you of $\alpha(x) \vee \beta(x)$.

Great so the OR is true. But you don't know which is true: $\alpha(x)$ OR $\beta(x)$.

So you try cases. By way of cases assume $\alpha(x)$.

Now can you conclude $\forall y \ \alpha(y)$? I say no. Why? Because we were not really fair to the adversary. The adversary could have given you x' instead. And then maybe $\beta(x')$ might be true. That is what the hint was getting at. "Being an Or, there is no consistency. One day she may give you this and the next day that."

We know that for each x, either $\alpha(x)$ OR $\beta(x)$. But for different x it could be a different one. We are never going to get that for each x the same one is true. This is why the statement is not a tautology.

• Answer: Here is the faulty formal proof mirroring the faulty game.

1) Deduction Goal: $LHS \rightarrow RHS$				
2)	$\forall x \; (\alpha(x) \lor \beta(x))$	Assumption/Premise		
3)	$\alpha(x) \lor \beta(x)$	Remove \forall with free variable x		
4)	Goal $(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))$	by cases (3)		
5)	lpha(x)	Assumption by cases		
6)	$\forall x \; lpha(x)$	Add \forall		
7)	$\forall y lpha(y)$	Same		
8)	$(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))$	Build OR		
5)	eta(x)	Assumption by cases		
6)	$\forall x \ \beta(x)$	$\mathrm{Add} \; \forall$		
7)	$\forall z \ \beta(z)$	Same		
8)	$(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))$	Build OR		
9)	$(\forall y \ \alpha(y)) \lor (\forall z \ \beta(z))$	Proof by cases $(3,8,8)$		
12) L	$\mu HS \to RHS$	Conclude deduction.		

What is wrong with this proof? Oh no!!! We spent very little time on it.

1) Deduction Cool IIIC > DIIC

5)	$\alpha(x_{\rightarrow})$	Assumption by cases
6)	NO $\forall x \ \alpha(x)$	Can't add \forall

Here x_{\rightarrow} is not a free variable but arbitrary thing that is constant within the assumption.

3. (40 marks) Prove that if sequence $f = f(0), f(1), f(2), f(3), \ldots$ converges to c_f and sequence $g = g(0), g(1), g(2), g(3), \ldots$ converges to c_g , then sequence $f+g = [f+g](0), [f+g](1), [f+g](2), [f+g](3), \ldots$ converges to $c_{\langle f+g \rangle} = c_f + c_g$. Note that [f+g](i) = f(i) + g(i). Hint: Assume that oracle f assures you that

 $\exists c_f \ \forall \epsilon_f > 0 \ \exists n_{\langle 0, f \rangle} \ \forall n_f \geq n_{\langle 0, f \rangle} \ |f(n_f) - c_f| \leq \epsilon_f.$ Assume that oracle g assures you that

 $\exists c_g \ \forall \epsilon_g > 0 \ \exists n_{\langle 0,g \rangle} \ \forall n_g \geq n_{\langle 0,g \rangle} \ |g(n_g) - c_g| \leq \epsilon_g.$

Then by playing Jeff's prover/adversary game, you prove

 $\exists c_{\langle f+g \rangle} \; \forall \epsilon_{\langle f+g \rangle} > 0 \; \exists n_{\langle 0,f+g \rangle} \; \forall n_{\langle f+g \rangle} \geq n_{\langle 0,f+g \rangle} \; |[f+g](n_{\langle f+g \rangle}) - c_{\langle f+g \rangle}| \leq \epsilon_{\langle f+g \rangle}.$

Hints for the game proof:

- Be clear who is giving whom which values.
- All the subscripts x_{\exists} and y_{\forall} can be dropped if you want.
- $n_{\langle 0, f+q \rangle}$ is just the name of a variable.
- Adding values, cutting them in half, taking their max, and keeping them unchanged are fun.
- Answer: The proof here is the minimal needed for the marks. The longer one below adds more explanation for people trying to understand.

We are working over the reals. Assume a worst functions f and g given by an adversary.

To prove $F \wedge G \rightarrow RHS$, assume we have an oracle f to assure us of F and another g to assure us of G. Our goal is to prove the RHS.

I get from oracle f the value c_f and from oracle g the value c_g and from these I construct $c_{\langle f+g \rangle} = c_f + c_g$.

Let $\epsilon_{\langle f+g \rangle}$ be an arbitrary value (given to me by my adversary). I give both oracles the value $\frac{1}{2}\epsilon_{\langle f+g \rangle}$.

I get from oracle f the value $n_{\langle 0,f\rangle}$ and from oracle g the value $n_{\langle 0,g\rangle}$ and from these I construct $n_{\langle 0,f+g\rangle} = Max(n_{\langle 0,f\rangle}, n_{\langle 0,g\rangle}).$

Let $n_{\langle f+g \rangle} \ge n_{\langle 0,f+g \rangle}$ be an arbitrary value (given to me by my adversary). Being bigger than both $n_{\langle 0,f \rangle}$ and $n_{\langle 0,g \rangle}$, I can give this value to both oracles.

The oracles assure me that $|f(n_f) - c_f| \le \epsilon_f$ and that $|g(n_g) - c_g| \le \epsilon_g$.

I conclude with $|[f+g](n_{\langle f+g \rangle}) - c_{\langle f+g \rangle}| = |[f(n_f) + g(n_g)] - [c_f + c_g]| = |[f(n_f) - c_f] + [g(n_g) - c_g]| \le |f(n_f) - c_f| + |g(n_g) - c_g| \le \epsilon_f + \epsilon_g = \epsilon_{\langle f+g \rangle}.$ Done.

We are working over the reals. Assume a worst functions f and g given by an adversary.

To prove $F \wedge G \rightarrow RHS$, assume we have an oracle f to assure us of F and another g to assure us of G. Our goal is to prove the RHS.

In order to prove $\exists c_{\langle f+g \rangle}$ for which something is true, I have to construct a value for which it is true. To do this I get help.

Oracle f, who assures me that $\exists c_f$, gives me such a value c_f . Oracle g gives me c_g .

I construct $c_{\langle f+g \rangle} = c_f + c_g$.

To prove $\forall \epsilon_{\langle f+q \rangle} > 0$, I let $\epsilon_{\langle f+q \rangle} > 0$ be an arbitrary value given to me by the adversary.

Oracle f assures me that $\forall \epsilon_f > 0$ something is true. Hence, I am able to give her the value $\epsilon_f = \frac{1}{2} \epsilon_{\langle f+g \rangle}$ and she assures me that it is true for this value. Similarly, I give $\epsilon_g = \frac{1}{2} \epsilon_{\langle f+g \rangle}$ to oracle g.

To prove $\exists n_{(0,f+q)}$, I have to construct such a value. To do this I get help.

Oracle f, who assures me that $\exists n_{(0,f)}$, gives me such a value $n_{(0,f)}$. Oracle g gives me n_g .

I construct $n_{\langle 0, f+g \rangle} = Max(n_{\langle 0, f \rangle}, n_{\langle 0, g \rangle}).$

To prove $\forall n_{\langle f+g \rangle} \ge n_{\langle 0,f+g \rangle}$, I let $n_{\langle f+g \rangle} \ge n_{\langle 0,f+g \rangle}$ be an arbitrary value given to me by the adversary.

Oracle f assures me that $\forall n_f \geq n_{\langle 0,f \rangle}$ something is true. Hence, I am able to give her this same value $n_f = n_{\langle f+g \rangle}$ and she assures me it is true for this value. Note that as needed for my contract with her we have that $n_{\langle f+g \rangle} \geq n_{\langle 0,f+g \rangle} = Max(n_{\langle 0,f \rangle}, n_{\langle 0,g \rangle}) \geq n_{\langle 0,f \rangle}$. Similarly, I give $n_g = n_{\langle f+g \rangle}$ to oracle g.

To complete the game, I must prove $|[f+g](n_{\langle f+g \rangle}) - c_{\langle f+g \rangle}| \le \epsilon_{\langle f+g \rangle}$. Again, I ask the oracles. Oracle f assures me that $|f(n_f) - c_f| \le \epsilon_f$ as does oracle g. We conclude as follows.

I conclude with $|[f+g](n_{\langle f+g \rangle}) - c_{\langle f+g \rangle}| = |[f(n_f) + g(n_g)] - [c_f + c_g]| = |[f(n_f) - c_f] + [g(n_g) - c_g]| \le |f(n_f) - c_f| + |g(n_g) - c_g| \le \epsilon_f + \epsilon_g = \epsilon_{\langle f+g \rangle}.$ This completes the proof.

- Answer: Here is the formal proof.
 - 1) Deduction Goal: $[F \land G] \rightarrow RHS$ 2) $F \wedge G$ Assumption/Premise 3) $\exists c_f \ \forall \epsilon_f > 0 \ \exists n_{\langle 0, f \rangle} \ \forall n_f \geq n_{\langle 0, f \rangle} \ |f(n_f) - c_f| \leq \epsilon_f$ Separating And $\forall \epsilon_f \! > \! 0 \; \exists n_{\langle 0, f \rangle} \; \forall n_f \! \geq \! n_{\langle 0, f \rangle} \; |f(n_f) \! - \! c_{\langle f, \exists \rangle}| \! \leq \! \epsilon_f$ Remove \exists 4)5) $\exists n_{\langle 0,f\rangle} \; \forall n_f \geq n_{\langle 0,f\rangle} \; \left| f(n_f) - c_{\langle f,\exists \rangle} \right| \leq \frac{1}{2} \epsilon$ Remove \forall with value $\frac{1}{2}\epsilon$ 6) $\forall n_f \ge n_{\langle 0, f, \exists \rangle}(\epsilon) |f(n_f) - c_{\langle f, \exists \rangle}| \le \frac{1}{2}\epsilon$ Remove \exists (depends on ϵ) $|f(n) - c_{\langle f, \exists \rangle}| \leq \frac{1}{2}\epsilon$ 7)Remove \forall with value n $|g(n) - c_{\langle g, \exists \rangle}| \leq \frac{1}{2}\epsilon$ 8) Repeat lines 3-7 but for qLet $c_{\langle f+g,\exists\rangle} = c_{\langle f,\exists\rangle} + c_{\langle g,\exists\rangle}$ 9)10) $|[f+g](n) - c_{\langle f + g, \exists \rangle}| = |[f(n) + g(n)] - [c_{\langle f, \exists \rangle} + c_{\langle g, \exists \rangle}]|$ $= |[f(n) - c_{\langle f, \exists \rangle}] + [g(n) - c_{\langle g, \exists \rangle}]| \le |f(n) - c_{\langle f, \exists \rangle}| + |g(n) - c_{\langle g, \exists \rangle}| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ Note that $n \ge n_{\langle 0, f, \exists \rangle}(\epsilon)$ and $n \ge n_{\langle 0, g, \exists \rangle}(\epsilon)$ 11)and hence $n \ge n_{\langle 0, f+q, \exists \rangle}(\epsilon) = Max(n_{\langle 0, f, \exists \rangle}(\epsilon), n_{\langle 0, q, \exists \rangle}(\epsilon))$ $\forall n \geq n_{\langle 0, f+g, \exists \rangle}(\epsilon) |[f+g](n) - c_{\langle f+g, \exists \rangle}| \leq \epsilon$ 12)Add \forall (10) $\begin{array}{l} \exists n_{\langle 0,f+g \rangle} \forall n \geq n_{\langle 0,f+g \rangle} \mid [f+g](n) - c_{\langle f+g, \exists \rangle} \mid \leq \epsilon \\ \forall \epsilon > 0 \ \exists n_{\langle 0,f+g \rangle} \forall n \geq n_{\langle 0,f+g \rangle} \mid [f+g](n) - c_{\langle f+g, \exists \rangle} \mid \leq \epsilon \\ \exists c_{\langle f+g, \exists \rangle} \forall \epsilon > 0 \ \exists n_{\langle 0,f+g \rangle} \forall n \geq n_{\langle 0,f+g \rangle} \mid [f+g](n) - c_{\langle f+g, \exists \rangle} \mid \leq \epsilon \\ \exists c_{\langle f+g, \exists \rangle} \forall \epsilon > 0 \ \exists n_{\langle 0,f+g \rangle} \forall n \geq n_{\langle 0,f+g \rangle} \mid [f+g](n) - c_{\langle f+g, \exists \rangle} \mid \leq \epsilon \\ \end{cases}$ Add \exists 13)14)Add ∀ Add \exists 15)16) $[F \land G] \rightarrow RHS$ Conclude deduction.