EECS 1090 – Test 1 (2025)

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1. (10 marks) Find all possible assignments of the variables that makes the following expression true/satisfied. How many are there? Explain all of the steps in your search for the assignment and in proving that this assignment works. Use Purple table reasoning, not a table.
   $[w \rightarrow \neg w] \wedge [(x \vee \neg x) \rightarrow w]$.

   - Answer: The AND between the clauses means that each of them needs to be true.

     Clause $w \rightarrow \neg w$: Let's do proof by cases.
     If $w = T$, then $w \rightarrow \neg w$ gives that $w = F$. This is a contradiction.
     If $w = F$, then $w \rightarrow \neg w$ is automatically true.
     Hence, w is forced to be false.
     Another technique is to translate $\alpha \rightarrow \beta$ into $\neg\alpha \vee \beta$ and $w \rightarrow \neg w$ into $\neg w \vee \neg w$. This forces $w = F$.

     Clause $(x \vee \neg x) \rightarrow w$: $x \vee \neg x$ is true by exclusive middle no matter what x is. Modus ponens forces $w = T$.

     Contradiction: Above w was forced both to false and to true.

     In conclusion, $x =?$, and $w = F \wedge T$, makes each clause true. Because the values of x is not forced, but there is a contradiction with the value of w, there are $2 \times 0 = 0$ satisfying assignments.

2. (10+10=20 marks) Models

   (a) Suppose I give you the statement $\forall x \; 1+x = x+1 > x$ but purposely I do not specify which model/universe we are using. Who then chooses the model and what is the entire list of things that he gets to choose?

   - Answer: The adversary chooses the model by specifying:
     (a) The universe of objects that variables x represent;
     (b) What do $+$, 1, and $>$ mean?
     (c) He does not get to change the definition of $=$. In our logic, this always means that the two objects are the same.

   (b) What is the difference between a statement being true and being valid/tautology? Are the following true and/or a valid/tautology? What is the difference? Give a counter example.
   - $\forall x \; \alpha(x) \rightarrow \exists x \; \alpha(x)$
   - $\forall x \; x{+}1{>}x$

   - Answer: A statement is only true or false once a model has been specified. It is valid/tautology if it is true under every model/universe.
     The first statement is valid, because it is true no matter what the objects x are or how α is defined.
     The second statement is NOT true if we define $+$ to mean marriage, 1 to mean the devil, and $>$ to mean better, because it is not better for x to be married to the devil than to be alone.

3. (10+10+15=35 marks) For each of the following, either use the purple table to prove it valid or prove it is not.

   (a) From $A \vee B$, $A \rightarrow C$, and $B \rightarrow D$, conclude $C \wedge D$.

   - Answer: This is not always true. For a counter example, set A to be true and B false. Then by modus ponens, C needs to be true but D does not. As such, $C \wedge D$ is false.

   (b) From $A \vee B$, $A \rightarrow C$, and $B \rightarrow D$, conclude $C \vee D$.

   - Answer: This is valid.

1) $A \vee B$	Axiom
2) $A \rightarrow C$	Axiom

3) $B \to D$ Axiom
4) Cases Goal $C \vee D$ Cases A and B
5) Case A Assumption
6) C Modus Ponens (5&2)
7) $C \vee D$ Building OR (6)
8) Case B Assumption
9) D Modus Ponens (8&3)
10) $C \vee D$ Building OR (9)
100) $C \vee D$ Cases Conclusion (1,7 & 10)

(c) Use the purple table to prove the following statement:
$[\neg L \to (\neg G \vee \neg H)] \;\wedge\; [(H \wedge L) \to (F \wedge B)] \;\to\; [\neg B \to (\neg G \vee \neg H)]$.
Hint: This may look hard but you can just bang it out. Start by putting at the bottom of your page the line 100) $[\alpha \wedge \beta] \to RHS$??. To prove $\to$, use Deduction. If you have a $\to$, try to use Modus Ponens. If you don't quite have what you want, build it up with either the Build And or the Build OR rule. If you have an $C \vee D$ that you don't know what to do with, start a Proof By Cases of what you want from these cases. Number your lines so that you can refer to them when you use some rule. Do NOT convert the $\to$ into *and* or *or* or use the distributive rule.

- Answer:

1) Deduction Goal: $[\alpha \wedge \beta] \to RHS$
2) $\alpha \;\wedge\; \beta$ Assumption/Premise
3) $\neg L \to (\neg G \vee \neg H)$ Separating And
4) $(H \wedge L) \to (F \wedge B)$ Separating And
5) Deduction Goal: $\neg B \to (\neg G \vee \neg H)$
6) $\neg B$ Assumption/Premise
7) $\neg(F \wedge B)$ Building And (6)
8) $\neg(F \wedge B) \to \neg(H \wedge L)$ Contra Positive (4)
9) $\neg(H \wedge L)$ Modus Ponens (7&8)
10) $\neg H \vee \neg L$ De Morgan's Law (10)
11) Cases Goal $\neg G \vee \neg H$ Cases $\neg H$ and $\neg L$
12) Case $\neg H$ Assumption
13) $\neg G \vee \neg H$ Building OR (12)
14) Case $\neg L$ Assumption
15) $\neg G \vee \neg H$ Modus Ponens (15&3)
98 $\neg G \vee \neg H$ Cases Conclusion (10,13 & 15)
99) $\neg B \to (\neg G \vee \neg H)$ Conclude deduction.
100) $[\alpha \wedge \beta] \to RHS$ Conclude deduction.

4. (10+10+15 =35 marks) For each of the following, either prove that its true over the reals by playing Jeff's prover/adversary game or take the negation of it and prove that.

(a) $\forall a\ \exists y\ \forall x\ x \cdot (y + a) = 0$

- Answer: True: Let a be arbitrary. Let $y = -a$. Let x be arbitrary. $x \cdot (y+a) = x \cdot (-a+a) = x \cdot 0 = 0$.

(b) $\exists x,\ \forall y,\ y+x>2y$

- Answer: False: $\forall x,\ \exists y,\ y+x \leq 2y$ is true. Let x be arbitrary. Let $y = x + 1$. $y + x = y + (y - 1) \leq 2y$.

(c) We say that the sequence $f = f(0), f(1), f(2), f(3), \ldots$ converges if
$\exists c\ \forall \epsilon > 0\ \exists n_0\ \forall n \geq n_0\ |f(n) - c| \leq \epsilon$.
Does the sequence $f = 1, -1, 1, -1, \ldots$, i.e., $f(n) = (-1)^n$ converges or not.
Hint: your proof should have two cases. In the first case, a given value is positive and in the

second, it is negative. By symmetry, you only have to prove the second case.
Hint: $\frac{1}{3}$ and even numbers are great.

- Answer: It does not converge but osculates. The negation is $\forall c\ \exists \epsilon > 0\ \forall n_0\ \exists n \geq n_0\ |f(n) - c| > \epsilon$.
  Prove true.
  Let $c_\forall$ be arbitrary.
  As hinted, we can assume by symmetry, that $c_\forall \leq 0$.
  Let $\epsilon_\exists$ be $\frac{1}{3}$.
  Let n_0 be arbitrary.
  Let $n \geq n_0$ be the next even integer at least n_0.
  By definition, $f(n) = (-1)^n = 1$.
  We have that $|f(n) - c_\forall| \geq |1-0| > \frac{1}{3} = \epsilon$.