EECS 1019/1090 – Propositional Logic Practice

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Not to be handed in.

T is for *True*, F is for *False*. $\neg$ is *Not*, $\wedge$ is A for *And*, $\vee$ is *OR*, $\rightarrow$ is *Implies*, $\oplus$ is *Parity*,
$\forall$ is A for *forAll*, $\exists$ is E for *Exists*.

Each such rule (in the purple table) has the form: From α & β conclude γ.
- If you already have lines in your proof of the form α & β,
- then you can add the line of the form γ to your proof.

Recall, a proof is a sequence of statements, where each statement is
- either an axiom, i.e., known to be true
- or follows from previous lines using some rule from the purple table (or the book).

Number the lines of your proof 1, 2, 3, ...
For each, give the name of the rule you use to prove that line and the line numbers of any previous lines used.
Do not skip steps. (Except for dropping $\neg\neg$)
Be sure to indent appropriately.

Rule of inference	Name in Text	Name in Purple Table
p $p \rightarrow q$ $\therefore\ q$	Modus ponens	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore\ \neg p$	Modus tollens	Contra Positive + Modus Ponens
$p \rightarrow q$ $q \rightarrow r$ $\therefore\ p \rightarrow r$	Hypothetical syllogism	Transitivity
$p \vee q$ $\neg p$ $\therefore\ q$	Disjunctive syllogism	Selecting Or
p $\therefore\ p \vee q$	Addition	Eval/Building Or
$p \wedge q$ $\therefore\ p$	Simplification	Separating And
p q $\therefore\ p \wedge q$	Conjunction	Eval/Building And
$p \vee q$ $\neg p \vee r$ $\therefore\ q \vee r$	Resolution	
$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ $\therefore\ r$		Cases
$p \rightarrow q$ $\therefore\ \neg q \rightarrow \neg p$ $\therefore\ \neg p \vee q$		Contrapositive
$\neg(p \wedge q)$ $\therefore\ \neg p \vee \neg p$		De Morgan

1. Multiple Choice. Which sentence relates best to the given English?

   (a) Lumber, together with marijuana, are big exports: a) $p \wedge q$; b) $p \vee q$; c) $p \oplus q$; d) other

   (b) Would that be fries or salad?: a) $p \wedge q$; b) $p \vee q$; c) $p \oplus q$; d) other

   (c) x: a) T/F variable; b) T/F sentence; c) object; d) other

   (d) p: a) T/F variable; b) T/F sentence; c) object; d) other

   (e) α: a) T/F variable; b) T/F sentence; c) object; d) other

   (f) Converse of $p \rightarrow q$: a) $\neg p \rightarrow \neg q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; d) a & b; e) other

   (g) Contrapositive of $p \rightarrow q$: a) $\neg p \rightarrow \neg q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; d) a & b; e) other

   (h) Image of $p \rightarrow q$: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (i) p is sufficient for q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (j) p whenever q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (k) p is great with q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (l) p follows from q: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (m) p only if q. I read this one as a threat "You can have desert only if you eat your spinach." Which answer feels the most like this threat? a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

   (n) q is necessary for p: a) $p \rightarrow q$; b) $q \rightarrow p$; c) $\neg q \rightarrow \neg p$; e) other

2. Explain each of the following.

   (a) Tautology:

      i. What is the definition of the word *tautology*?

      ii. If you made a table with a row for each T/F assignment of the variables p, r, and q, how many rows would there be.
      Hint: I don't what you to check each of them.

(b) Deduction

(c) Separating And

(d) Proof by cases
Hint: Our cases will be "p is true" and "p is false. What is your γ?
Hint: Be sure that you prove all three steps needed for this rule before concluding.

(e) Excluded Middle

(f) Selecting Or

(g) Building/Eval Or

3. Let p, q, s and t denote propositional (true/false) variables. Find an assignment of truth vales to p, q, s, and t that makes the following expression true. Justify your answer.
$$[p \leftrightarrow q] \ \wedge\ [(s \leftrightarrow \neg s) \to (p \vee q)] \ \wedge\ [(\neg t) \to s] \ \wedge\ [\neg(t \vee p)]$$

4. $(x = 2 \text{ or } x = 5) \to (x-2)(x-5) = 0$

5. $[(\alpha \to \beta) \text{ and } (\neg\beta \vee \gamma)] \to [\neg\gamma \to \neg\alpha]$

6. The goal is to translate any truth table for a Boolean formula/sentence into Disjunctive Normal Form (DNF).
Such a sentence is the $\vee$/OR of many clauses.
Each such clause is the $\wedge$/AND of many literals.
Each such literal is either a variable or its negation.
Eg. $(A \wedge \neg B \wedge \neg C) \vee (B \wedge E \wedge F)$.

(a) Each row of the truth table, gives the evaluation of the sentence under a given an assignment A. Such an assignment gives T/F value to each of the variables. Construct a clause that says "The variables have assignment A". Denote this clause with $clause(A)$. For example, what would clause $clause(A)$ be that is equivalent to stating the assignment is $A = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = T) \rangle$?

(b) Given a truth table for sentence S, let $S_T = \{A \mid A \text{ satisfies } S\}$ be the set of assignments A under which formula S evaluates to be true, i.e., the assignment could be A_1 OR A_2 OR .... Here each satisfying assignments in S_T is listed. For example, $S = p_1 \oplus p_2$ is satisfied iff exactly one of the variables is true, i.e., the assignment is $A_1 = \langle (p_1 = T) \wedge (p_2 = F) \rangle$ or is $A_2 = \langle (p_1 = F) \wedge (p_2 = T) \rangle$. Explain how to form a DNF expression for a general sentence S. For example, what is it for the specific sentence $S = p_1 \oplus p_2$.

(c) Consider $p_1 \oplus p_2 \oplus p_3 \oplus \ldots \oplus p_n$. For which of the T/F assignments is this true? What is this sentence called?

(d) How many clauses would its full DNF have?

(e) Consider the equivalence $(\alpha \wedge p) \vee (\alpha \wedge \neg p) \equiv \alpha$. It collapses the two clauses into one with the variable p removed. Note how if α is satisfied, then the variable p can flip between T and F. Use the rules in the purple table to prove the

(f) Suppose there are two satisfying assignments/clauses that are the same for all variables, except the value of one of the variables is flipped. For example, suppose the sentence S is satisfied with both the assignment $A_1 = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = T) \rangle$ and $A_2 = \langle (p_1 = T) \wedge (p_2 = F) \wedge (p_3 = F) \rangle$. How can you use the previous question to collapse/merge these into one equivalent clause?

(g) Suppose the sentence S is $p_1 \vee p_2$. It has three satisfying assignments, $A_1 = \langle (p_1 = T) \wedge (p_2 = T) \rangle$, $A_2 = \langle (p_1 = T) \wedge (p_2 = F) \rangle$, and $A_3 = \langle (p_1 = F) \wedge (p_2 = T) \rangle$. What do these clauses merge into? Hint: one clause can merge with more than one other clause.

(h) Suppose you have an assignment A that satisfies sentence $S = p_1 \oplus p_2 \oplus p_3 \oplus \ldots \oplus p_n$. If you keep all the variables fixed except for one, and flip the value of the remaining variable, what happens to the resulting value of S? Can any of the clauses of S collapse?

7. Proofs using Purple table:

(a) Your mother insists that you either put out the garbage or do the dishes. You convince her that you have put out the garbage and run out the door. State the rule used and its name (in purple table or in book)

(b) If you put out the garbage, your mother will be happy. If you do the dishes, your mother will be happy. Prove that if (you put out the garbage or you do the dishes), your mother will be happy. Prove this about *garbage*, *dishes* and *happy*.
Hint: My proof uses two rules in the purple table and 11 lines.

8. $[A \rightarrow B] \rightarrow [B \rightarrow A]$

   (a) If valid, i.e.. true in every setting, what would it mean?

   (b) In class, Jeff demonstrates that this is not always true by giving a counter example involving objects in our daily life. Give that example or, if you can't remember it, make up another.

   (c) Give an assignment to A and B under which this expression evaluates to false.
   Do this by forming a tree of T/F. Under the sentence below, write T or F under each variable. Below that write T or F for each [...]. Below this write F for the entire expression.
   $[A \rightarrow B] \rightarrow [B \rightarrow A]$

   (d) There is a rule that translates $[X \rightarrow Y]$ into the OR/$\vee$ expression $[??\vee??]$.
   Use this rule to translate each of the $\rightarrow$ into Or/$\vee$.
   Then use other rules to put it into the easier of:
   - Conjunctive Normal Form $(A \vee \neg B \vee \neg C) \wedge (\neg D \vee E \vee F)$
   - Disjunctive Normal Form $(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E \wedge F)$.
   You might want to check that under the setting from the previous question, the statement is still false.

9. *Resolution* plays an important role in AI and is used in Prologue.
   It's *Cut Rule* corresponds to the statement: $S \equiv [[(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]]$.

   (a) Prove S.
   Hint: My proof uses the following rules in the given order. (a) Deduction (b) Separating And (c) Proof by cases (d) Excluded Middle (e) Selecting Or (f) Building/Eval Or
   Hint: My proof has 14 lines.
   Hint: Our cases will be "p is true" and "p is false. What is your γ?

   (b) Now that you have proved that $[(\neg p \vee r) \wedge (p \vee q)] \rightarrow [r \vee q]$ is true, you added to the Jeff's big purple table of rules. How then do you use it in a proof.

10. Simplifying Parity (Purple Table): Suppose I tell you that α is true. I claim that this means that everywhere $\alpha \oplus \beta$ appears it can be simplified to $\neg\beta$. Let's consider whether this is true.

    (a) What does $\alpha \oplus \beta$ NOT means?

      i. At least one of these is true.
      ii. Exactly one of these is true.
      iii. We add them together in binary where 1+1=0.
      iv. $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$.
      v. All correct meanings.

    (b) Assuming α, which of the following is NOT true?

      i. $\alpha \oplus \beta$ is true iff β is false.
      ii. β is true then $\alpha \oplus \beta$ is false.
      iii. β must be true.
      iv. $\alpha \oplus \beta$ iff $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$.

v. All true

(c) Prove $\alpha \to [(\alpha \oplus \beta) \equiv \neg\beta]$.

(d) Prove $\alpha \to [\neg\beta \to [(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)]]$.

(e) Prove $\alpha \to [[(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)] \to \neg\beta]$.