EECS 1090 – Oracle Game Problems

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*** I banged this out with global search and replace from

`https://www.eecs.yorku.ca/~jeff/courses/1090/ppt/1090-9-Quesions.pptx`

Let me know if there are typos.
Hint: Order of operations gives:
  $\forall x\ A \rightarrow B$ is $(\forall x A) \rightarrow B$
  $\forall x\ A\ \&\ B$ is $(\forall x A)\ \&\ B$
  $A\ \&\ B \rightarrow C$ is $(A \& B) \rightarrow C$
  $A\ \&\ B \vee C$ is $(A \& B) \vee C$
Hint: All of the questions require you to:
  Remove the quantifiers using either
    The Oracle−Prover−Adversary Game
    The Formal Rules
  Do some simple Purple Table Rule
  Add the quantifiers back
A common Purple Table Rule is transitivity, i.e.
  Assume $A \rightarrow B$
  Assume $B \rightarrow C$
  Prove $A \rightarrow C$
Hint: See the solution for 2i in the slides.
This solution can be read either as an Oracle−Prover−Adversary Game or as Formal Rules.
Use (Oracle−Prover−Adversary Game or Formal Proof) & Purple Table to prove each of the following.
Sorry, I don't have any other solutions.

1. Some:
   (a) $\neg\forall y\ (F(y)\ \&\ G(y))$
       $\rightarrow \exists y\ (\neg F(y) \vee \neg G(y))$
   (b) $\forall w\ (L(w) \rightarrow M(w))$
       $\&\ \forall y\ (M(y) \rightarrow N(y))$
       $\rightarrow \forall w\ (L(w) \rightarrow N(w))$
   (c) $\exists x\ (G(x)\ \&\ A(x))$
       $\&\ \forall y\ (C(y) \rightarrow \neg G(y))$
       $\rightarrow \exists z (A(z)\ \&\ \neg C(z))$
   (d) $\neg\exists x\ (\neg R(x)\ \&\ S(x,x))$
       $\&\ S(j,j)$
       $\rightarrow R(j)$
   (e) $\forall x\ ((\neg C(x,b) \vee H(x)) \rightarrow I(x,x))$
       $\&\ \exists y\ \neg I(y,y)$
       $\rightarrow \exists x\ C(x,b)$
   (f) $\forall x\ F(x)$
       $\&\ \forall z\ H(z)$
       $\rightarrow \neg\exists y\ (\neg F(y) \vee \neg H(y))$

2. More:
   (a) $\forall x\ \neg J(x)$
       $\&\ (\exists y\ (H(y) \vee R(y,y))) \rightarrow \exists x\ J(x)$
       $\rightarrow \forall y\ \neg(H(y) \vee R(y,y))$
   (b) $\neg\exists x\ \forall y\ (P(x,y)\ \&\ \neg Q(x,y))$
       $\rightarrow \forall x\ \exists y\ (P(x,y) \rightarrow Q(x,y))$
   (c) $\forall x\ \neg(\forall y\ (H(y,x) \vee T(x)))$
       $\&\ \neg\exists y\ (T(y) \vee \exists x\ \neg H(x,y))$
       $\rightarrow \forall x\ \forall y\ H(x,y)\ \ \&\ \ \forall x\ \neg T(x)$

(d) $\forall z\ (L(z) \text{ iff } H(z))$
$\&\ \forall x\ \neg(H(x) \vee \neg B(x))$
$\rightarrow \neg L(b)$

(e) $\forall z\ [K(z) \rightarrow (M(z)\ \&\ N(z))]$
$\&\ \exists z\ \neg N(z)$
$\rightarrow \exists x\ \neg K(x,x)$

(f) $\exists x\ (\neg B(x,m)\ \&\ \forall y\ (C(y) \rightarrow \neg G(x,y)))$
$\&\ \forall z\ (\neg\forall y\ (W(y) \rightarrow G(z,y)) \rightarrow B(z,m))$
$\rightarrow \forall x\ (C(x) \rightarrow \neg W(x))$

(g) $\exists z\ Q(z) \rightarrow \forall w\ (L(w,w) \rightarrow \neg H(w))$
$\&\ \exists x\ B(x) \rightarrow \forall y\ (A(y) \rightarrow H(y))$
$\rightarrow \exists w\ (Q(w)\ \&\ B(w)) \rightarrow \forall y\ (L(y,y) \rightarrow \neg A(y))$

(h) $\forall y\ (K(b,y) \rightarrow \neg H(y))$
$\rightarrow\ \forall x\ [\exists y\ (K(b,y)\ \&\ Q(x,y)) \rightarrow \exists z\ (\neg Hz\ \&\ Q(x,z))]$

(i) Solution in slides
$\neg\forall x\ (\neg G(x) \vee \neg H(x)) \rightarrow \forall x\ (C(x)\ \&\ \forall y\ (L(y) \rightarrow A(x,y)))$
$\&\ \exists x\ [H(x)\ \&\ \forall y\ (L(y) \rightarrow A(x,y))] \rightarrow \forall x\ (F(x)\ \&\ \forall y\ B(x,y))$
$\rightarrow (\neg\forall x\ \forall y\ B(x,y)) \rightarrow \forall x\ (\neg G(x) \vee \neg H(x))$

3. Even More:

(a) $\forall x\ (A(x) \rightarrow B(x))$
$\rightarrow \forall x\ (B(x) \vee \neg A(x))$

(b) $\forall x\ (A(x) \rightarrow (A(x) \rightarrow B(x)))$
$\rightarrow \forall x\ (A(x) \rightarrow B(x))$

(c) $\neg\exists x\ (A(x) \vee B(x))$
$\rightarrow \forall x\ \neg A(x)$

(d) $\forall x\ (A(x) \rightarrow B(x))$
$\vee\ \exists x\ A(x)$

(e) $(\exists x\ A(x) \rightarrow \exists x\ B(x))$
$\rightarrow \exists x\ (A(x) \rightarrow B(x))$

(f) $\forall x\ \exists y\ (A(x) \vee B(y))$
iff $\exists y\ \forall x\ (A(x) \vee B(y))$

4. Equivalent Statements:

(a) $\neg\forall x\ (A(x) \rightarrow B(x))$
iff $\exists x\ (A(x)\&\ \neg B(x))$

(b) $(\exists x\ \exists y\ A(x,y)) \rightarrow A(a,b)$
iff $(\exists x\ \exists y\ (A(x,y))$ iff $A(a,b))$

(c) $\neg\forall x\ \neg[(A(x)\ \&\ B(x)) \rightarrow C(x)]$
iff $\exists x\ [\neg A(x) \vee (\neg C(x) \rightarrow \neg B(x))]$

(d) $\neg\forall x\ \exists y\ [(A(x)\ \&\ B(x)) \vee C(y)]$
iff $\exists x\ \forall y\ [\neg(C(y) \vee A(x)) \vee \neg(C(y) \vee B(x))]$

(e) $\forall x\ (A(x) \text{ iff } B(x))$
iff $\neg\exists x\ [(\neg A(x) \vee \neg B(x))\ \&\ (A(x) \vee B(x))]$

(f) $\forall x\ (A(x)\ \&\ \exists y\ \neg B(x,y))$
iff $\neg\exists x\ [\neg A(x) \vee \forall y\ (B(x,y)\ \&\ B(x,y))]$

5. (Sorry. Not proof read) Show that each of the following sets of sentences is inconsistent.

(a) $[[\forall x\ (M(x) \text{ iff } [(x)) \ \&\ \neg Mc] \ \&\ \forall x\ ]$

(b) $[\neg Fa, \neg\exists x\ (\neg F(x) \vee \neg F(x))]$

(c) $[\forall x\ \forall y\ l(x,y) \to \neg\exists z\ T(z)$
$(V(x))(V(y))L(x,y)(((w))C(w,w)v(3(z))T(z))$
$(\neg\forall x\ \forall y\ L(x,y) \vee \forall z\ B(z,z)k)$
$(\neg(V(z))B(z,z)kv\neg(3)C(w,w))$
$\forall x\ \forall y\ [(x,y))$

(d) $[\exists x\ \forall y\ (H(x,y) \to \forall w\ \rfloor(w,w))$
$(3(x))\neg J(x,x)$
$\neg(7(x))\neg H(x)m)$

(e) $[\forall x\ \forall y\ (G(x,y) \to Hc)$
$(3(x))Gi(x)\ \&\ \sqrt{x})((y))(V(z))L(x,y,z), Lcib$
$v\neg(Hc \vee Hc)]$

(f) $[\forall x\ [(S(x)\ \&\ B(x,x)) \to Ka(x)]$
$(\sqrt{x})(H(x)B(x,x))$
$(3(x))(S(x)\ \&\ H(x)$
$\forall x\ \neg(Ka(x)\ \&\ H(x))|$