Math 1090-O Exam Winter 2023-2024

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Fill out the bubble sheets at the end of the exam.
Even if you have to guess, answer EVERY question.
If you guess a True/False question, you will get half right.
As such, 50% may or may not be a passing grade.
It will depend on how other people do.
There are $3 \times 60/104 = 1.5$ minutes per question.
Be sure to hand in the question pages as well as the bouble sheets
(I will know and be unhappy if you don't)

Warm Up

1. (a)T(b)F: Assume that "All snakes are reptiles" and that "Some reptiles hatch their eggs themselves". We can we conclude "Some snakes hatch their eggs themselves?"

2. (a)T(b)F: Assume that "All snakes are reptiles", that "All reptiles are animals and lay eggs", and that "Many snakes only have one lung". Can we conclude "Some animals that lay eggs only have one lung."

3. (a)T(b)F: The truth of a math statement in a given model in our logic might depend on one's perspective or on the day.

Purple Table

4. (a)T(b)F: Stating that $\alpha \to \beta$ eliminates the possibility that we are in a universe in which α is true and β is false.

5. (a)T(b)F: When proving $\alpha \to \beta$ by deduction, we assume that α is true because we know that it is.

6. (a)T(b)F: When proving $\alpha \to \beta$ by deduction, we assume that α is true because the case in which α is false is trivial.

7. (a)T(b)F: If a propositional math statement (ie one with $\wedge$, $\vee$, and $\to$) appears as line in a sound proof, then the statement evaluates to be true in given every setting of the variables for which any given the axioms are true.

8. (a)T(b)F: $\alpha \to \beta$ means that α being true causes β to be true.

9. (a)T(b)F: If α is false, then $\alpha \to \beta$ is automatically false and hence $\neg(\alpha \to \beta)$ can be added as a line of your proof.

10. (a)T(b)F: If β is false, then $\alpha \to \beta$ is automatically false and hence $\neg(\alpha \to \beta)$ can be added as a line of your proof.

11. (a)T(b)F: If β is true, then $\alpha \to \beta$ is automatically true and hence $(\alpha \to \beta)$ can be added as a line of your proof.

12. (a)T(b)F: Modus Ponens, i.e., $\alpha \& (\alpha \to \beta)$ gives β is a rule which says that if both α and $\alpha \to \beta$ are earlier lines of your proof, then you can also include β as a line of your proof.

13. (a)T(b)F: Modus Ponens is a sound rule because no matter which True/False values α and β have, if both α and $\alpha \to \beta$ evaluate to be true, then β also evaluates to true.

14. (a)T(b)F: The English statement "α is sufficient for β" is translated into $\alpha \to \beta$.

15. (a)T(b)F: The English statement "α is necessary for β" is translated into $\alpha \to \beta$ because if α is not true than β is not true.

16. (a)T(b)F: Proof by cases states that if you have already proved both $\alpha \to \gamma$ and $\beta \to \gamma$, then you can conclude that γ is true.

17. (a)T(b)F: Building/Eval OR states that if you have already proved α, then you can conclude that $\alpha \vee \beta$ is true even if you do not know whether β is true or false.

18. (a)T(b)F: Building/Eval OR states that if you have already proved $\neg\alpha$, then you can conclude that $\neg(\alpha \vee \beta)$ is true even if you do not know whether β is true or false.

19. (a)T(b)F: From $\alpha \to \beta$, it follows that $\beta \to \alpha$.

20. (a)T(b)F: Selecting Or states that if both α and $\alpha \vee \beta$ are earlier lines of your proof, then you can also include β as a line of your proof.

21. (a)T(b)F: γ *and* $(\alpha$ *or* $\beta)$ iff $(\gamma$ *and* $\alpha)$ *or* $(\gamma$ *and* $\beta)$

22. (a)T(b)F: γ *and* $(\alpha$ *or* $\beta)$ iff $(\gamma$ *or* $\alpha)$ *and* $(\gamma$ *or* $\beta)$

23. (a)T(b)F: γ *or* $(\alpha$ *and* $\beta)$ iff $(\gamma$ *or* $\alpha)$ *and* $(\gamma$ *or* $\beta)$

24. (a)T(b)F: The statement $(P \wedge Q) \to (P \vee Q)$ is a tautology.

25. $(\alpha \to \gamma)$ *or* $(\beta \to \gamma)$ is equivalent to:
(a) $(\alpha$ *or* $\beta) \to \gamma$;
(b) $(\alpha$ *and* $\beta) \to \gamma$;
(c) $\gamma \to (\alpha$ *or* $\beta)$;
(c) $\gamma \to (\alpha$ *and* $\beta)$;
(e) other

26. If we set α to be true then $(\alpha \to \beta)$ *and* $(\neg\alpha \to \gamma)$ simplifies to:
(a) β; (b) $\neg\beta$; (c) γ; (d) $\neg\gamma$; (e) β *or* γ

27. If set γ to be true then $(\alpha \to \gamma)$ *and* $(\beta \to \neg\gamma)$ simplifies to:
(a) β; (b) $\neg\beta$; (c) γ; (d) $\neg\gamma$; (e) β *or* γ

Counting Assignments

28. The number of different assignments to the three variables $\langle P, Q, R\rangle$, for example $\langle true, false, true\rangle$ is:
(a) 3^2; (b) 2^3; (c) 3×2; (d) 3; (e) other;

29. The number of different assignments $\langle P, Q, R\rangle$ in which $(P \wedge Q) \wedge (P \to Q)$ is true is:
(a) 1; (b) 2; (c) 3; (d) 6; (e) other;

30. The number of different assignments $\langle P, Q, R\rangle$ in which $(P \vee Q) \wedge (\neg R)$ is true is:
(a) 1; (b) 2; (c) 3; (d) 6; (e) other;

31. The number of different assignments $\langle P, Q, R\rangle$ in which $(P \vee Q)$ is true is:
(a) 1; (b) 2; (c) 3; (d) 6; (e) other;

32. The number of different assignments $\langle P, Q, R\rangle$ in which $P \wedge (P \vee Q)$ is true is:
(a) 1; (b) 2; (c) 3; (d) 6; (e) other;

33. The number of different assignments $\langle P, Q, R\rangle$ in which $\neg P \wedge (P \vee Q)$ is true is:
(a) 1; (b) 2; (c) 3; (d) 6; (e) other;

34. The number of different assignments of the TWO variables $\langle P, Q\rangle$ in which $(P \vee Q) \to (P \wedge Q)$ is FALSE is:
(a) 0; (b) 1; (c) 2; (d) 3; (e) 4;

35. The number of different assignments of the TWO variables $\langle P, Q\rangle$ in which $(P \wedge Q) \to (P \vee Q)$ is FALSE is:
(a) 0; (b) 1; (c) 2; (d) 3; (e) 4;

Models

36. In this course, a model/universe specifies many things. Which of the following does it NOT specify:
(a) the universe of objects that variables x represent;
(b) for each predicate α and each object x whether or not $\alpha(x)$ is true;
(c) for each function f and each object x which object $f(x)$ maps to;
(d) The value to every free variable;
(e) all of the above are specified by the model.

37. Which are NOT examples of a *term*: (a) x; (b) $f(x)$; (c) $\alpha(x)$; (d) $x + y$; (e) all of the above are terms

38. (a)T(b)F: If a specific model like the Integers has not been specified, we start the proof by says "Consider an arbitrary model given to us by an adversary."

39. (a)T(b)F: If a specific model like the Integers has not been specified, our goal is to prove that a given math statement is true.

40. (a)T(b)F: If a specific model like the Integers has been specified, our goal is to prove that a given math statement is true in that model.

41. (a)T(b)F: A math statement is said to be valid if it is true in some model.

42. (a)T(b)F: The induction statement $[S(0) \& \forall i\ S(i-1) \rightarrow S(i)] \rightarrow [\forall i\ S(i)]$ is true in every model.

43. (a)T(b)F: The statement $1+1=2$ is true in every model.

44. (a)T(b)F: We did not cover it, but computers manipulate bits which operate in the model called Boolean Algebra. In this model, $1+1=0$.

45. (a)T(b)F: The statement $\alpha \wedge \beta = \beta \wedge \alpha$ is true in every model that we would ever consider.

46. (a)T(b)F: To prove a math statement is not valid, we need to find one model in which it is not true.

47. Which is the first universe below for which $\exists$ *politician* p $\forall$ *voters* v *Loves*(v,p) is false:
(a) Jeff is universally loved;
(b) America is split into those that love Trump and those that love Biden;
(c) Everyone has a favorite;
(d) What? I hate them all;
(e) It is true in all of them

48. Which is the first universe below for which $\forall$ *voters* v $\exists$ *politician* p *Loves*(v,p) is false:
(a) Jeff is universally loved;
(b) America is split into those that love Trump and those that love Biden;
(c) Everyone has a favorite;
(d) What? I hate them all;
(e) It is true in all of them

49. Consider the reals. Which is the first functions f below for which $\exists y\ \forall x\ y = f(x)$ is false:
(a) $f(x) = 5$;
(b) $f(x) = x^2$;
(c) $f(x) = \sqrt{x}$;
(d) It is true in all of them

50. Consider the reals. Which is the first function f below for which $\forall x\ \exists y\ y = f(x)$ is false:
(a) $f(x) = 5$;
(b) $f(x) = x^2$;
(c) $f(x) = \sqrt{x}$;
(d) It is true in all of them

51. Consider the reals. Which is the first function f below for which $\forall y\ \exists x\ y = f(x)$ is false:
(a) $f(x) = 2x$;
(b) $f(x) = x^3$;
(c) $f(x) = 5$;
(d) $f(x) = \sqrt{x}$;
(e) It is true in all of them

52. Consider the reals. Which is the first function f below for which $\exists x\ \forall y\ y = f(x)$ is false:
(a) $f(x) = 5$;
(b) $f(x) = x^2$;
(c) $f(x) = \sqrt{x}$;
(d) It is true in all of them

53. Which lines are NOT equivalent to $\neg[\forall x\ \exists y \le x\ \alpha(x,y)]$:
(a) $\exists x\ \neg[\exists y \le x\ \alpha(x,y)]$;
(b) $\exists x\ \forall y > x\ \neg\alpha(x,y)$;
(c) $\exists x\ \forall y \le x\ \neg\alpha(x,y)$;
(d) They are all equivalent

English/Valid

For the next 4 quesions consider the following English meanings:
(a) There is an additive zero.
(b) Every real has an additive inverse.
(c) Except for one (zero), every real has a multiplicative inverse.
(d) There are fractions
(e) It is not valid

54. $\forall x\ \exists y\ x{+}y{=}0$.

55. $\exists y\ \forall x\ x{+}y{=}x$.

56. $\exists a\ \forall x\ \exists y\ [x = a \text{ and } x \cdot y = 5]$.

57. $\forall y,\ \exists x,\ y{=}2x{+}1$.

58. (a)T(b)F: $\forall a\ \exists y\ \forall x\ x \cdot (y + a) = 0$

59. (a)T(b)F: $[\forall x_1\ \exists y_1\ \forall x_2\ \exists y_2\ P(x_1, y_1, x_2, y_2)] \Rightarrow [\forall x_1\ \forall x_2\ \exists y_1\ \exists y_2\ P(x_1, y_1, x_2, y_2)]$

60. (a)T(b)F: $[\forall x_1\ \exists y_1\ \forall x_2\ \exists y_2\ P(x_1, y_1, x_2, y_2)] \Rightarrow [\exists y_1\ \forall x_1\ \exists y_2\ \forall x_2\ P(x_1, y_1, x_2, y_2)]$

61. (a)T(b)F: $\alpha(0) \to [\forall x\ \alpha(x)]$ is valid.

62. (a)T(b)F: $\exists y\ [\alpha(y) \to [\forall x \alpha(x)]]$ is valid.

63. (a)T(b)F: $\forall y\ [\alpha(y) \to [\exists x \alpha(x)]]$ is valid.

64. (a)T(b)F: $\exists y\ \alpha(y)$ is a possibly infinite OR.

Proof

65. Where does the oracle come from? Which of the following statements is false.
(a) When proving $A \to B$, proof by deduction assumes A and then goes on to prove B. While we are assuming A, we can pretend that we have an oracle that assures us that A is true.
(b) If A is an axiom that we are assuming is true, then we can pretend that we have an oracle that assures us that A is true.
(c) Jeff made this up. I doubt George would like it.
(d) The oracle is a burning bush on top of a mountain that can answer questions for you.
(e) All of the above are true

For the next 4 quesions consider the following options:
(a) Prover;
(b) Adversary;
(c) Oracle;
(d) Jeff;
(e) Someone else

66. For the prover to prove $\exists x\ \alpha(x)$ who provides the object x?

67. For the prover to prove $\forall x\ \alpha(x)$ who provides the object x?

68. For the oracle to assure $\exists x\ \alpha(x)$ who provides the object x.

69. For the oracle to assure $\forall x\ \alpha(x)$ who provides the object x?

For the next 4 quesions consider the following options:
(a) We first prove $\alpha(t)$ for some term t and then add the $\exists x$ giving the required line. The corresponding English would be "Because the previous line assures us that the statement is true for the proof's constructed value t, we know that there exists a value of x for which it is true.
(b) We first prove $\alpha(x)$ for some free variable x and then add the $\forall x$ giving the required line. Note the adversary chooses the value of x. The corresponding English would be "Because the previous line assures us that the statement is true for whatever object the free variable x represents, we know that it is true for all values of x.
(c) The next line of the proof can be the statement $\alpha(x_\exists)$ with the $\exists x$ removed and the x replaced with $x_\exists$. The corresponding English would be "Let $x_\exists$ denote the object which the previous line assures exists."
(d) The next line of the proof can be the statement $\alpha(t)$ with the $\forall x$ removed and the x replaced with some term t constructed by the prover. The corresponding English would be "Because the previous line assures us that the statement is true for every object x, we know it is true for proof's constructed value t."
(e) None of the above.

70. Suppose in a formal proof you have already proved or assumed $\forall x\ \alpha(x)$. How is this used?

71. What steps are taken to formally prove $\forall x\ \alpha(x)$?

72. Suppose in a formal proof you have already proved or assumed $\exists x\ \alpha(x)$. How is this used?

73. What steps are taken to formally prove $\exists x\ \alpha(x)$?

For the next 4 quesions consider the following proof:

1) Deduction Goal: $\alpha(x) \rightarrow \exists x \alpha(x)$		
2)	$\alpha(x)$	Assumption/Premise
3)	$\exists x \alpha(x)$	Add $\exists$
4) $\alpha(x) \rightarrow \exists x \alpha(x)$		Conclude deduction.

74. (a)T(b)F: The proof can't be sound because the statement proved is not valid.

75. (a)T(b)F: Line 2 is wrong. When assuming things an arrow $x_\rightarrow$ needs to be added.

76. (a)T(b)F: Line 3 is wrong. You can't add $\exists x$ to $x_\rightarrow$.

77. (a)T(b)F: You can't add $\forall x$ to $x_\rightarrow$.

Minimum Value

For the next few quesions consider the following defintion:
Let $S^+ = \{x \in Reals \mid x > 0\}$ denote the set of positive reals.

78. Which is true about the minimum value in S^+?
(a) It is like 0.00001 but has an infinite description.
(b) It is zero.
(c) It does not exist.
(d) It is 1.

79. Which of these say something different?
(a) $\exists x \in S^+,\ \forall y \in S^+, x \leq y$.
(b) $\forall x \in S^+, \exists y \in S^+, y < x$.
(c) x is the minimum in S^+.
(d) x is in S^+ and other values in S^+ are bigger.
(e) They all say the same thing.

80. What is the first line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
(a) Let x be 0.
(b) Let x be an arbitrary value in S^+.
(c) Let x be the minimum value in S^+.
(d) By way of contradiction, assume $x \geq y$.
(e) Let y be an arbitrary value in S^+.

81. What is the second line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
(a) Let $y = \frac{1}{2}x$,
(b) Let y be 0.
(c) Let y be the minimum value in S^+.
(d) Let x be an arbitrary value in S^+.
(e) Let $y = x + 1$.

82. What is the third line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
(a) This gives a contradiction.
(b) y is the minimum value in S^+.
(c) y is not the minimum value in S^+.
(d) x is the minimum value in S^+.
(e) $y \in S^+$ and $y < x$.

83. What is the fourth line in a proof of $\forall x \in S^+, \exists y \in S^+, y < x$?
(a) Hence x is the minimum value in S^+.
(b) Hence by way of contradiction, the opposite must be true.
(c) Hence x (and all values) is not the minimum value in S^+.
(d) This has nothing to do with minimums.

Parsing

84. When informally using the prover/adverary/oracal game proving $\exists x\ [\alpha(x) \rightarrow \beta(x)]$, first step is:
(a) Let x be an arbitrary value given by the adversary.
(b) Deduction, i.e., assume LHS and prove RHS.
(c) None of the above.

85. When informally using the prover/adverary/oracal game proving $[\forall x\ \alpha(x)] \rightarrow [\exists y\ \alpha(y)]$, first step is:
(a) Let x be an arbitrary value given by the adversary.
(b) Let y be a specific value constructed by the prover.
(c) The oracle assures us that the LHS is true and the prover proves the RHS.
(d) None of the above.

86. When formally proving the previous statement, the top level proof technique is:
(a) Remove the $\exists x$.
(b) Add the $\exists x$.
(c) Deduction, i.e., assume LHS and prove RHS.
(d) Let x be an arbitrary value given by the adversary.
(e) None of the above.

Induction

87. (a)T(b)F: The induction axiom is $[S(0) \& \forall i (S(i-1) \rightarrow S(i))] \rightarrow [\forall i S(i)]$.

88. (a)T(b)F: The induction axiom could also be $[S(0) \& \forall i (S(i) \rightarrow S(i+1))] \rightarrow [\forall i S(i)]$.

89. (a)T(b)F: The induction axiom is an axiom because it is true in every possible model.

90. (a)T(b)F: The induction axiom is the same as the Alcoholics Anonymous motto - One day at a time. It is sufficient that each day you manage not to drink for one more day. If you can get out of bed in the morning and at each point are able to take the next step, then you can travel the entire road.

91. (a)T(b)F: You don't have to solve $S(i)$ alone. You should use the oracle's assurance that $S(i-1)$ is true.

92. (a)T(b)F: With strong induction, the oracle assures you that that $S(0), S(1), S(2), \ldots, S(i\text{–}1)$ are true.

93. (a)T(b)F: One of Jeff's religions is not to worry about the entire iterative algorithm all at once. First determine how to establish the loop invariant. Then determine how to maintain it while making progress. Then determine how to obtain the postcondition from the loop invariant and the exit condition.

94. (a)T(b)F: The loop invariant is the action taken within the loop.

95. (a)T(b)F: One of Jeff's religions is not to worry about the entire recursive algorithm all at once. You can give any instances to your friends solve as long as they meet the precondition and are smaller. Do not micro manage your friends. If the input is sufficiently small solve it yourself.

96. (a)T(b)F: To understand a recursive algorithm, one needs to be able to trace it the entire computation in detail.

Computation

97. Which statements are FALSE about the following:
$\forall\ Inputs\ x,\ \exists\ Java\ Program\ J,\ J(x) = P(x)$
(a) It states that computational problem P is computable;
(b) Jeff want you to feel physical pain when you read it;
(c) It is true for every computational problem P (even uncomputable ones);
(d) They are all true.

98. Which statements are TRUE about the following:
$\exists\ Java\ Program\ J,\ \forall\ Inputs\ x,\ J(x) = P(x)$
(a) It states that computational problem P is computable;
(b) Jeff want you to feel physical pain when you read it;
(c) It is true for every computational problem P (even uncomputable ones);
(d) They are all false.

99. (a)T(b)F: $\exists A, \forall I, \exists c, A(I) = P(I) \& Time(A, I) \leq |I|^c$ states that computational problem P is computable in polynomial time.

100. (a)T(b)F: $\forall I, \exists A, \neg Halting(I) = A(I)$

101. (a)T(b)F: $\exists P, \forall A, \exists I, A(I) \neq P(I)$

102. (a)T(b)F: $\forall k, \exists A, \forall I, Lines(A, I) = k$ where $lines(A, I)$ says that on input I algorithm A has k lines of code. (Yes, I know that the number of lines of code should not depend on the input. That is the whole point of the question.)

103. (a)T(b)F: I say that k is chosen at compile time.

104. (a)T(b)F: $\forall k, \exists A, \forall I, Lines(A, I) = k \& A(I) = Sorting(I)$. Here k could be 0 or a billion.